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Model Categories

with a view towards

Rational Homotopy Theory

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Abstract

This thesis is an investigation into the theory of model categories with applications in the foundations of rational homotopy theory. For a model category $\mathcal{C}$ we construct the associated homotopy category $\mathrm{Ho}(\mathcal{C})$ and prove that this construction yields a localization of $\mathcal{C}$ with respect to the class of weak equivalences. We also prove a total derived functor theorem. This gives sufficient conditions for a functor between two model categories to induce an equivalence of categories between the associated homotopy categories. We then turn our attention to some specific examples of model categories. We put a model structure on the category of r-reduced simplicial sets and r-reduced simplicial groups. In both cases the weak equivalences are those maps inducing isomorphism on rational homotopy groups. We then prove a general result which, given a category $\mathcal{C}$ (satisfying some conditions), provides a model structure on the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$. This theorem is applied to give model structures on the category of r-reduced simplicial complete rational Hopf algebras and the r-reduced simplicial rational Lie algebras, respectively. Finally we put a model structure on the category of r-reduced differential graded rational Lie algebras, where the weak equivalences are the maps inducing isomorphism on homology. As we prove the model category axioms for the various structures we also construct pairs of adjoint functors between them and show that these satisfy the conditions of the total derived functor theorem. As a result, the homotopy category $\mathrm{Ho}(\mathbf{sSet}_1^{\mathbb{Q}})$ of 1-reduced simplicial sets is equivalent to the homotopy category $\mathrm{Ho}(\mathbf{dgLie}_0)$ of connected differential graded Lie algebras.

Resumé

Dette speciale omhandler modelkategoriteori med applikationer til grundlaget for rationel homotopiteori. Givet en modelkategori $\mathcal{C}$ konstruerer vi den associerede homotopikategori $\mathrm{Ho}(\mathcal{C})$ og viser at denne konstruktion giver en lokalisering af $\mathcal{C}$ ved de svage ækvivalenser. Vi beviser også en sætning om totaltderiverede funktorer. Denne giver tilstrækkelige betingelser for at en funktor mellem to modelkategorier inducerer en ækvivalens mellem de associerede homotopikategorier. Dernæst vender vi vores opmærksomhed mod nogle specifikke eksempler på modelkategorier. Vi lægger en modelstruktur på kategorien af r-reducerede simplicielle mængder og r-reducerede simplicielle grupper. I begge tilfælde er de svage ækvivalenser de afbildninger som inducerer isomorfier på de rationelle homotopigrupper. Dernæst viser vi et generelt resultat der, givet en kategori $\mathcal{C}$ (som opfylder nogle bestemte betingelser), lægger en modelstruktur på kategorien $s\mathcal{C}$ af simplicielle objekter i $\mathcal{C}$. Dette teorem anvendes så til at lægge modelstrukturer på kategorien af r-reducerede simplicielle fuldstændige rationelle Hopf-algebraer og kategorien af r-reducerede simplicielle rationelle Lie-algebraer. Til sidst lægger vi en modelstruktur på kategorien af r-reducerede differentialgraduerede rationelle Lie-algebraer, hvor de svage ækvivalenser er de afbildninger som inducerer isomorfier på homologi. Samtidig med at vi viser modelkategoriaksiomerne for de forskellige strukturer, konstruerer vi parvist adjungerede funktorer mellem kategorierne, og viser at disse opfylder betingelserne for sætningen om totaltderiverede funktorer. Af dette følger at homotopikategorien $\mathrm{Ho}(\mathbf{sSet}_1^{\mathbb{Q}})$ af 1-reducerede simplicielle mængder er ækvivalent med homotopikategorien $\mathrm{Ho}(\mathbf{dgLie}_0)$ af sammenhængende differentialgraduerede Lie-algebraer.

Contents

Preface

The intended reader of this thesis is a graduate student who has some background in algebraic topology. I assume basic knowledge about simplicial sets including the basics of simplicial homotopy theory.

The contents of the thesis come from several different sources; see the bibliography for references. At the beginning of each chapter (and some sections) I have included a short description of the sources used in that chapter (or section).

I have tried to follow the advice of R. P. Boas on the art of exposition of mathematics, namely that "long convoluted sentences, bifurcating into a plethora of dependent clauses, especially those with verbs deferred to the end, with the consequent effect of demanding close attention from the reader, as well as comprehension of sesquipedalian and abstruse words, or of highly specialized technical jargon, are rebarbative and should be sedulously avoided"[1].

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[1]This quote is from *Lion Hunting and other Mathematical Pursuits* edited by G. L. Alexanderson and D.H. Mugler, MAA 1995.

Chapter 1

Introduction

In 1967 Daniel Quillen published the paper [Qui67] thereby founding *homotopical algebra*. Quillen describes homotopical algebra as the "generalization of homological algebra to arbitrary categories which results by considering a simplicial object as being a generalization of a chain complex".
The work of John Milnor ([Mil57]) and Daniel Kan ([Kan58]) in the late 50s showed that the "ordinary" pointed homotopy theory of topological spaces was in fact equivalent to the "homotopy theory" of reduced simplicial sets and to the "homotopy theory" of simplicial groups. Therefore there was a need for a general categorical framework which could explain the nature of these equivalences. In [Qui67] Quillen introduced such a framework with his definition of *model categories*. Model categories have since become the central notion of the homotopical algebra. Quillen explains that the term "model category" is short for "a category of models for a homotopy theory". Indeed, the same homotopy theory may have many different models, as the work by Milnor and Kan has shown.
In 1969 ([Qui69]) Quillen demonstrated the powerful ideas of [Qui67] by showing that the rational homotopy theory of simply-connected pointed spaces is equivalent to the homotopy theory of reduced differential graded Lie algebras. This is no small feat. Indeed just proving the axioms of a model category can be quite difficult. The reward for doing this is the definite homotopical content thus produced. For example, Quillen uses the quoted equivalence to derived several new spectral sequences for rational homotopy theory and to solve problems posed by both Heinz Hopf and René Thom. For more about the history of rational homotopy theory and Quillen's role in its development see [Hes99] and [TVP13].
This thesis aims at studying the theory of model categories as it is exemplified in [Qui69]. Quillen establishes his equivalence by five steps, starting with the category of spaces, going through several simplicial categories and ending at the category of differential graded Lie algebras. The structure of the thesis is based around these five steps. We will examine each of the categories and show how to put the corresponding model category structure on the category.

This first chapter serves as a quick introduction to Quillen's strategy in [Qui69]. This should motivate the detailed considerations in the later chapter. In particular we go through Quillen's Lie model construction $\lambda : \mathbf{Top}_1 \to \mathbf{dgLie}_1$ which associates to a simply connected space, X, a differential graded Lie algebra, λX, carrying "the same" rational homotopy theoretical information as X.

Homotopy groups

In classical homotopy theory one studies topological spaces by examining their homotopy groups. For a space X, the *n-th homotopy group* (where $n \geq 1$) of X is the set $[S^n, X]$ of homotopy classes of continuous based maps $S^n \to X$. The set $[S^n, X]$ may be equipped with a group structure since S^n is a co-group object in the homotopy category. For $n \geq 2$ there are n different group structures on $[S^n, X]$. Though different, the multiplication in one group structure is a group-homomorphism with respect to any other structure. By the "Eckmann-Hilton argument" it follows that these structures all coincide and are commutative. The resulting Abelian group is denoted $\pi_n X$.
For $n = 1$ the Eckmann-Hilton argument cannot be applied since there is only one group structure on $[S^1, X]$, thus the first homotopy group $\pi_1 X$ may be non-Abelian. This group is usually called the *fundamental group* of the space X. For $n = 0$ the space $[S^0, X]$ of homotopy classes of *based* maps is in bijection with the path-components of X. This set is denoted $\pi_0 X$. A space such that $\pi_0 X = *$ is a point, is called *path connected*. A connected space, whose fundamental group is trivial, is called *simply connected*. In a simply connected space all pairs of points may be connected by a continuous path in the space in exactly one way, up to homotopy equivalence.
A central problem in homotopy theory is the calculation of the homotopy groups of spheres S^k. In more detail, one wants to have explicit descriptions of the groups $\pi_n S^k$ for various n and k. For example it can be shown that $\pi_1 S^1 \cong \mathbb{Z}$ and $\pi_n S^1 = 0$ for $n \geq 2$. In general this problem is very open.
In the 1950s Serre proved the following result.

Theorem 1.0.1. *(Serre) The Abelian groups $\pi_n S^k$ for $n \geq 2$ are all finitely generated.*

In fact Serre gave a more specific description of $\pi_n S^k$. He showed that $\pi_n S^k$ is finite except when $n = k$ or when k is even and $n = 2k - 1$. For $n = k$ and $n = 2k - 1$ the group $\pi_n S^k$ has rank 1. In other words the study of the homotopy groups of spheres is really the study of the torsion components of these groups. If one disregards torsion, then the problem is solved by Serre's result.
Since CW-complexes are built out of spheres and discs one might wonder if the study of homotopy groups of general spaces might also be solvable if one disregards torsion. Indeed this might apply to all spaces since any space is weakly

equivalent to a CW-complex, i.e. for a space X there is a CW-complex QX and a map $QX \to X$ which induces isomorphisms on homotopy groups.
One useful way of “disregarding torsion” in an Abelian group A is to tensor with $\mathbb{Q}$, the ring of rational numbers. This has the added bonus of not only creating a torsion free Abelian group $A \otimes_{\mathbb{Z}} \mathbb{Q}$, but in fact creates a rational vector space. Of course this really relies on the assumption that A is Abelian. Thus we must restrict our attention to simply connected spaces.
At this point the story merges with a different story about homotopy groups. It turns out that for any space X there is a natural product structure on the tower $\pi_* X$ of homotopy groups. This is the *Whitehead product* $[,] : \pi_p X \times \pi_q X \to \pi_{p+q-1} X$. The Whitehead product is $\mathbb{Z}$-bilinear, graded commutative and satisfies the graded Jacobi identity. It is thus very close to being a graded Lie algebra. Considering instead of $\pi_* \Omega X$, the rational homotopy groups $\pi_* \Omega X \otimes \mathbb{Q}$ we get a fully fledged reduced graded Lie algebra over $\mathbb{Q}$, called the *rational homotopy Lie algebra* of X. This provides a functor $\pi_*(\Omega -) \otimes \mathbb{Q}$ from the category of simply connected spaces, to the category of reduced graded Lie algebras over $\mathbb{Q}$. Thus from spaces we get graded Lie algebras. The famous topologist, Heinz Hopf, had asked the following question.

Question 1. Does every reduced graded Lie algebra over $\mathbb{Q}$ occur as the rational homotopy Lie algebra of some simply connected space?

Graded Lie algebras also appear when taking the homology of a *differential* graded Lie algebra. Thus it is natural to ask the following question.

Question 2. Does there exist a functor $\lambda : \mathbf{Top}_1 \to \mathbf{dgLie}_0$ between the category of simply connected spaces and reduced differential graded Lie algebras such that the homology of λX coincides with the rational homotopy Lie algebra $\pi_* X \otimes \mathbb{Q}$?

Quillen provided such a functor λ and used it to answer Hopf's question in the affirmative.

The construction of λ

We now sketch Quillen's construction of the functor $\lambda : \mathbf{Top}_1 \to \mathbf{dgLie}_0$. The functor λ is realized as the composition of several functors, each of which we briefly sketch below. We are given a simply connected space X and will associate, in a functorial way, a reduced differential graded Lie algebra.

From spaces to simplicial sets

At the time Quillen wrote his paper it was well known that studying the homotopy theory of spaces was equivalent – in the sense of model categories, more about this later – to studying the homotopy theory of simplicial sets. The functors

establishing this equivalence are Sing : **Top** $\to$ **sSet**, the *singular complex*, and its left adjoint $|\cdot|$: **sSet** $\to$ **Top** the *geometric realization*.
Thus $\mathrm{Sing}(X)$ provides a simplicial model for the given space X. However we want a model that takes into account our knowledge that X is simply-connected. For this we apply the *Eilenberg subcomplex* functor $E_1 : \mathbf{sSet} \to \mathbf{sSet}_1$ which takes a pointed simplicial set K and removes those simplices whose 1-skeleton is not at the basepoint. The result is then a 1-reduced simplicial set E_1K. If K is a Kan complex then the inclusion $E_1K \to K$ is a weak equivalence of simplicial sets. Since $\mathrm{Sing}(X)$ is a Kan complex it follows that $E_1\mathrm{Sing}X$ is a sensible 1-reduced simplicial model for X. Taking the geometric realization $|E_1\mathrm{Sing}X|$ yields a space which is weakly homotopic to X. This gives the first pair of adjoints $|\cdot| \dashv E_1\mathrm{Sing}(\cdot)$.

$$\mathbf{Top}_1 \underset{E_1\mathrm{Sing}}{\overset{|\cdot|}{\leftrightarrows}} \mathbf{sSet}_1$$

From simplicial sets to simplicial groups

In the passage to the simplicial world we have given ourselves a combinatorial model of the space X, namely the 1-reduced simplicial set $K = E_2\mathrm{Sing}X$. We now wish to look at the "space of loops" in X. Topologically we could just look at the loop space ΩX. But since we have shifted to the simplicial world we need to work a little to get good models of $\Omega(-)$. This work was done by Kan in [Kan58] who constructed a pair of adjoints

$$\mathbf{sSet}_0 \underset{\overline{W}}{\overset{G}{\rightleftarrows}} \mathbf{sGrp}$$

where GK is a model for the loop space of K and $\overline{W}A$ is a classifying space for the group A.
We apply G to our 1-reduced model $E_2\mathrm{Sing}X$ to get a reduced simplicial group $GE_2\mathrm{Sing}X$.

From groups to complete Hopf algebras

Given a group G, the rational group algebra $\mathbb{Q}G$ may be equipped with a coalgebra structure with the map $\Delta g = g \otimes g$ (for $g \in G$) as the coproduct. The coalgebra structure is compatible with the algebra structure and so $\mathbb{Q}G$ is a Hopf algebra. Given any Hopf algebra H the subspace of primitive elements (those elements x such that $\Delta x = x \otimes 1 + 1 \otimes x$) $\mathcal{P}H$ is a Lie algebra with the commutator as Lie bracket. However, for the group algebra $\mathbb{Q}G$ the subspace of primitive elements is trivial, so we need a more refined Hopf algebra in order to retain the homotopical

information. Quillen's solution is to complete $\mathbb{Q}G$ with respect to the filtration defined by powers of the augmentation ideal $\mathrm{Ker}(\varepsilon)$. Having completed $\mathbb{Q}G$ we get a complete Hopf algebra $\widehat{\mathbb{Q}}G$ where the exponential function $\exp : \widehat{\mathbb{Q}}G \to \widehat{\mathbb{Q}}G$ (defined by the usual power series formula) now makes sense. Furthermore for a primitive element x the exponential element $\exp(x)$ is group-like. Conversely, if x is group-like then the logarithm $\log(x)$ is primitive. Thus $\mathcal{P}\widehat{\mathbb{Q}}G$ might plausibly retain homotopical information about the original group G.
This whole strategy is carried out in each level of the reduced simplicial group $GE_2\mathrm{Sing}X$. This yields a simplicial complete Hopf algebra $\widehat{\mathbb{Q}}GE_2\mathrm{Sing}X$.
The "group-like elements" functor $\mathcal{G} : \mathbf{sCHopf} \to \mathbf{Grp}$ is right adjoint to the functor $\widehat{\mathbb{Q}}$.

$$\mathbf{sGrp}_0 \underset{\mathcal{G}}{\overset{\widehat{\mathbb{Q}}}{\rightleftarrows}} \mathbf{sCHopf}_0$$

Furthermore, for free simplicial groups G (such as the group $GE_2\mathrm{Sing}X$) Quillen proves that G is weakly rational homotopy equivalent to $\mathcal{G}\widehat{\mathbb{Q}}G$ (assuming G is connected and free), thus it again seems reasonable to suppose that not much homotopical information is lost in the passage $GE_2\mathrm{Sing}X \longmapsto \widehat{\mathbb{Q}}GE_2\mathrm{Sing}X$.

From simplicial complete Hopf algebras to simplicial Lie algebras

In the previous step we hinted strongly at how we might go from a (reduced simplical complete) Hopf algebra H to a reduced simplicial Lie algebra, namely by considering the subspace of primitive elements, with the commutator bracket. Thus, to our simplicial complete Hopf algebra $\widehat{\mathbb{Q}}GE_2\mathrm{Sing}X$ we associate the simplicial Lie algebra $\mathcal{P}\widehat{\mathbb{Q}}GE_2\mathrm{Sing}X$.
Again the operation of taking primitives is somewhat reversible. The left adjoin to $\mathcal{P}$ is given by the completed universal enveloping algebra functor $\widehat{U} : \mathbf{Lie} \to \mathbf{CHopf}$, where the completion is with respect to the augmentation ideal.

$$\mathbf{sCHopf}_0 \underset{\mathcal{P}}{\overset{\widehat{U}}{\rightleftarrows}} \mathbf{sLie}_0$$

Since $\mathcal{P}\widehat{\mathbb{Q}}GE_2\mathrm{Sing}X$ is reduced it is in particular connected. Given any connected free simplicial Lie algebra $\mathfrak{g}$, Quillen proves that $\mathfrak{g}$ is weakly equivalent to $\mathcal{P}\widehat{U}\mathfrak{g}$, thus it seems likely that we do not throw away too much homotopical information when passing from the Hopf algebra to the primitives.

From simplicial Lie algebras to dg Lie algebras

We arrive at the final step in Quillen's construction. At present we have a reduced simplicial Lie algebra $\mathfrak{g} = \mathcal{P}\widehat{\mathbb{Q}}GE_2\mathrm{Sing}X$. This is in particular a simplicial $\mathbb{Q}$-vector space, and the Dold-Kan correspondence tells us that these carry the same homotopy theory as the homotopy theory of non-negatively graded chain complexes over $\mathbb{Q}$ i.e. differential graded vector spaces over $\mathbb{Q}$. Thus the final step must be to prove a specific version of the Dold-Kan correspondence, linking simplicial Lie algebras with dg Lie algebras. This is exactly what Quillen does. The normalized chains functor $N : \mathbf{sVect} \to \mathbf{dgVect}$ can be restricted to a functor $N : \mathbf{sLie} \to \mathbf{dgLie}$. This works by taking the simplicial Lie brackets and shuffling them together (using the Eilenberg-Mac Lane map) to get a graded Lie bracket. Furthermore N has a left adjoint N^*

$$\mathbf{sLie}_0 \underset{N}{\overset{N^*}{\leftrightarrows}} \mathbf{dgLie}_0$$

Quillen's construction is therefore the composite

$$\lambda = N\mathcal{P}\widehat{\mathbb{Q}}GE_2\mathrm{Sing} : \mathbf{Top}_1 \to \mathbf{dgLie}_0.$$

Making sense of the construction

We shall view Quillen's construction through the lens of *model categories*; a conceptual homotopy-theoretic framework developed by Quillen a few years before rational homotopy theory.
The axioms of a model category have changed slightly since Quillen's first steps in [Qui67].

Definition 1.0.1. A *model category* is a complete and cocomplete category $\mathcal{C}$ equipped with three distinguished classes of morphisms; the *weak equivalences*, the *fibrations* and the *cofibrations*. A morphism in $\mathcal{C}$ is called an *acyclic fibration* if it is both a weak equivalence and a fibration, likewise an *acyclic cofibration* is both a cofibration and a weak equivalence. We require the following axioms be satisfied.

(i) The weak equivalences satisfy the 2-out-of-3 axiom.
(ii) All three classes are closed under retracts.
(iii) Acyclic cofibrations have the LLP with respect to fibrations. Acyclic fibrations have the RLP with respect to cofibrations.
(iv) Any morphism may be factored in two ways: as a cofibration followed by an acyclic fibration, or as an acyclic cofibration followed by a fibration.

The structure of a model category $\mathcal{C}$ allows one to define a sensible notion of *homotopy* between morphisms and so also a notion of *homotopy equivalence* between the objects of $\mathcal{C}$. The weak equivalences play the same role that classical weak equivalences (i.e. continuous maps inducing isomorphism on homotopy groups) do in topology. In particular a weak equivalence between cofibrant-fibrant objects is in fact a homotopy equivalence between these objects (Whitehead's theorem).

The notion of homotopy equivalence in a model category $\mathcal{C}$ allows one to define the associated homotopy category $\mathrm{Ho}(\mathcal{C})$ of $\mathcal{C}$. It turns out that this is a localization of $\mathcal{C}$ with respect to the weak equivalences, i.e. there is a canonical functor

$$\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$$

such that $\gamma(f)$ is an isomorphism whenever f is a weak equivalence, and γ is universal with this property.
The axioms of model categories are phrased in a language very reminiscent of the category of topological spaces. Indeed the category of topological spaces with continuous maps does carry a model structure (in fact several). However, this is by no means the only example. For example, each of the categories through which Quillen's λ-functor passes admits a model structure, as we explain below. First, here is a classical example which illustrates the flexibility of the model category axioms.

Theorem 1.0.2. *Let R be a ring and $\mathrm{Ch}(R)$ the category of chain complexes (non-negatively graded) of left R-modules. For a map $f : M \to N$ in $\mathrm{Ch}R$ we make the following definitions.*

(i) The map f is a weak equivalence *if it induces an isomorphism on homology (also known as a* quasi-isomorphism*)*
(ii) The map f is a fibration *if it is an epimorphism in positive degrees.*
(iii) The map f is a cofibration *if it is a monomorphism having projective cokernels in all degrees.*

With these definitions $\mathrm{Ch}(R)$ is a model category.

Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories and $F : \mathcal{C} \to \mathcal{D}$, $U : \mathcal{D} \to \mathcal{C}$ are adjoint functors with F left adjoint to U. We call F a *left Quillen functor* if F preserves cofibrations and acyclic cofibrations. Likewise U is a *right Quillen functor* if U preserves fibrations and acyclic fibrations. The adjunction (F, U) is called a Quillen adjunction if F is a left Quillen functor (equivalently if U is a right Quillen functor). A Quillen adjunction (F, U) is called a *Quillen equivalence* if the following condition is satisfied:

- For cofibrant A in $\mathcal{C}$ and fibrant X in $\mathcal{D}$ and every map $g : A \to UX$. The map g is a weak equivalence in $\mathcal{C}$ if and only if $g^\flat : FA \to X$ is a weak equivalence in $\mathcal{D}$.

The terminology is justified by the *Total derived functor theorem* which we now state.

Theorem 1.0.3. *Let (F, U) be a Quillen equivalence. Then the left derived functor* $\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ *exists, as does the right derived functor* $\mathbf{R}U : \mathrm{Ho}(\mathcal{D}) \to \mathrm{Ho}(\mathcal{C})$*. These functors form an adjoint equivalence of categories.*

Quillen uses Theorem 1.0.3 repeatedly to show that λ induces an equivalence on the homotopy level.

Chapter **2**

Model categories

The purpose of this chapter is to lay out the basic theory of model categories. Our exposition follows [Hov99], [DS95], [Hir03], and [GJ99].
In general [DS95] provides a quick introduction starting from scratch. They go through the model structure on the category of chain complex in some depth. [Hov99] is a good place to find detailed explanations of examples and [Hir03] is good for detailed axiomatics and thorough investigations of abstract model category theory. The material on model categories in [GJ99] is strongly inclined towards simplicial methods (as the title of their book suggests).

2.1 Model categories

Given a category $\mathcal{C}$ we let $\mathcal{C}^{\to}$ denote the associated *category of morphisms* whose objects are the morphisms of $\mathcal{C}$ and whose morphisms are commutative squares. A morphism f in $\mathcal{C}$ is a **retract** of a morphism g in $\mathcal{C}$ if f is a retract of g (in the usual sense) as objects of $\mathcal{C}^{\to}$. Explicitly this means that there is a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

where the horizontal compositions are identity morphisms.
Given morphisms $i : A \to B$ and $p : X \to Y$ in $\mathcal{C}$ then i has the **left lifting property** (LLP) **with respect to** p and p has the **right lifting property** (RLP) **with respect to** i if, for every commutative diagram (without h)

$$\begin{array}{ccc} A & \longrightarrow & C \\ \big\downarrow{\scriptstyle i} & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & D \end{array}$$

there exists a morphism $h : B \to X$ making the filled diagram commutative. Note that we do *not* require the filling morphism h to be unique.

Definition 2.1.1. A **model structure** on a category $\mathcal{C}$ consists of three specified subcategories: *weak equivalences* $\mathcal{W}$, *cofibrations* **Cofib**, and *fibrations* **Fib**. A morphims which is both a fibration and a weak equivalence is called an *acyclic fibration*, likewise a morphism which is both a cofibration and a weak equivalence is called an *acyclic cofibration*. We require the following axioms are satisfied for morphisms f and g of $\mathcal{C}$;

2-Out-of-3 Axiom If f and g are composable and two of $\{f, g, gf\}$ are weak equivalences, then so is the third.

Retract Axiom If f is a retract of g and g is a weak equivalence, cofibration or fibrations, then so is f.

Lifting Axiom Acyclic cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to acyclic fibrations.

Factorization Axiom Every morphism f has two factorizations

(i) $f = hi$ where i is a cofibration and h is an acyclic fibration.
(ii) $f = pj$ where p is a fibration and j is an acyclic cofibration.

Remark 2.1.1. When speaking of a weak equivalence, fibration or cofibration we mean a morphism in the respective subcategory, not an object.

Definition 2.1.2. A **model category** is a category $\mathcal{C}$ which has all small limits and colimits together with a model structure on $\mathcal{C}$.

We abuse notation by suppressing the distinguished subcategories ($\mathcal{W}$, **Cofib** and **Fib**) and will always write $\mathcal{C}$ for the model category.

Remark 2.1.2. Some authors require the factorization axiom to be functorial (for example [Hov99] and [Hir03]). Some authors (including Quillen in [Qui67] and [Qui69]) only require $\mathcal{C}$ to be finitely (co)complete.

Notation 1. We frequently use the standard notation: $\xrightarrow[\sim]{}$ for weak equivalences, $\hookrightarrow$ for cofibrations and $\twoheadrightarrow$ for fibrations.

2.2 Basic examples

Example 2.2.1 (Trivial structure). Let $\mathcal{C}$ be any complete and cocomplete category. Let the weak equivalences consist of all isomorphisms in $\mathcal{C}$, and let every map be both a fibration and a cofibration. With these choices $\mathcal{C}$ is a model category. In fact we get a model structure on $\mathcal{C}$ by choosing any one of $\mathcal{W}$, **Cofib** or **Fib** to be the subcategory of isomorphisms and letting the other two be all of $\mathcal{C}$.

Most often it takes some hard work to verify the model structure axioms. We therefore wait with the more advanced examples until later in this thesis. However it may be beneficial to look ahead at the Section 2.7 and Section 2.7 to get a feel for the classical examples of model structures.

Example 2.2.2 (Product structure). Given model categories $\mathcal{C}$ and $\mathcal{D}$, the product category $\mathcal{C} \times \mathcal{D}$ may be endowed with the *product model structure* where a morphism (f, g) belongs to one of the distinguished subcategories if and only if both f and g belong to the corresponding subcategory of $\mathcal{C}$ and $\mathcal{D}$, respectively.

Example 2.2.3 (Under). Suppose $\mathcal{C}$ is a model category and A an object of $\mathcal{C}$. The *under-category* $A/\mathcal{C}$ (objects are morphisms $A \to X$ out of A, morphisms are commutative triangles) is then a model category where a morphism is a weak equivalence, cofibration, or fibration if it is one in $\mathcal{C}$. Of course one must take care that colimits in $A/\mathcal{C}$ are not simply computed in $\mathcal{C}$.

Example 2.2.4 (Basepoint). Note the following special case of the under-category model structure; let $\mathcal{C}$ be a model category and $*$ the terminal object. The under-category $*/\mathcal{C}$ will then be denoted $\mathcal{C}_*$. Objects in this category may be thought of as objects X of $\mathcal{C}$ together with a specified "basepoint" $b : * \to X$. Then morphisms in $\mathcal{C}_*$ are morphisms in $\mathcal{C}$ that preserve the basepoint.

Example 2.2.5 (Over). If $\mathcal{C}$ is a model category and X an object of $\mathcal{C}$, then the *over-category* $\mathcal{C}/X$ (objects are morphisms $A \to X$ into X, morphisms are commutative triangles) is a model category where a morphism is a weak equivalence, cofibration, or fibration it it is one in $\mathcal{C}$. In this case one must be careful that limits are not simply computed in $\mathcal{C}$.

2.3 Basic properties

The model category axioms are self-dual in the sense that if $\mathcal{C}$ is a model category then the opposite category $\mathcal{C}^{op}$ has the opposite model structure where the cofibrations of $\mathcal{C}^{op}$ are the fibrations of $\mathcal{C}$, the fibrations of $\mathcal{C}^{op}$ are the cofibrations of $\mathcal{C}$ and the weak equivalences remain the same. This fact has the practical consequence that all theorems deduced directly from the model category axioms come in dual pairs. Thus having proven one theorem one automatically deduces the dual theorem.

We now state a number of useful results. The proofs are not hard and are omitted from this exposition.

Lemma 2.3.1 (The Retract Argument)**.** *Let $\mathcal{C}$ be any category and $f = pi$ a factorization of a morphism f in $\mathcal{C}$. If f has the LLP with respect to p then f is a retract of i. Dually, if f has the RLP with respect to i then f is a retract of p.*

Lemma 2.3.2. *Suppose $\mathcal{C}$ is a model category.*

(i) A morphism is a cofibration if and only if it has the LLP with respect to all acyclic fibrations.

(ii) A morphism is an acyclic cofibration if and only if it has the LLP with respect to all fibrations.

(iii) A morphism is a fibration if and only if it has the RLP with respect to all acyclic cofibrations.

(iv) A morphism is an acyclic fibration if and only if it has the RLP with respect to all cofibrations.

Corollary 2.3.1. *Any isomorphism in a model category is both an acyclic fibration and an acyclic cofibration.*

From Lemma 2.3.2 the weak equivalences together with the cofibrations determine the collection of fibrations. Dually, the weak equivalences together with the fibrations determine the cofibrations. In fact the collections of cofibrations and fibrations also determine the weak equivalences. To see this, let f be a morphism in the model category $\mathcal{C}$. By assumption f may be factored as $f = pj$ with p a fibration and j a map which has the LLP with respect to all fibrations. Now, by the 2-out-of-3 property, f is a weak equivalence if and only if p has the RLP with respect to all cofibrations. We thus have the following result.

Lemma 2.3.3. *Let $\mathcal{C}$ be a model category. Any two of the subcategories $\mathcal{W}$,* **Cofib** *and* **Fib** *determine the third.*

Lemma 2.3.4. *Suppose $\mathcal{C}$ is a model category. The cofibrations and acyclic cofibrations are closed under pushouts. Dually, the fibrations and acyclic fibrations are closed under pullbacks.*

Since model categories are assumed complete and cocomplete they have both a terminal object 0 and an initial object 1.

Definition 2.3.1. An object A in $\mathcal{C}$ is said to be **cofibrant** if the unique morphism $0 \to A$ is a cofibration. An object X in $\mathcal{C}$ is said to be **fibrant** if the unique morphism $X \to 1$ is a fibration.

2.4 Homotopy theory in model categories

In the category of topological spaces the usual definition of the homotopy relation between continuous maps $f, g : X \to Y$ requires the use of the unit interval $I = [0,1]$ to form the "cylinder on X", namely the product space $X \times I$. If $\mathcal{C}$ is a general category its objects may be very different from topological spaces, in particular there may be no obvious replacement for the unit interval in $\mathcal{C}$. In a model category, however, we can still carry out the cylinder construction. This is axiomatized in the following definition.

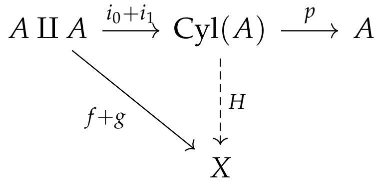
Definition 2.4.1. Suppose $\mathcal{C}$ is a model category and let $f, g : A \to X$ be morphisms in $\mathcal{C}$.

(i) A **cylinder object** for A is an object $\mathrm{Cyl}(A)$ and a factorization

$$A \amalg A \xrightarrow{i_0+i_1} \mathrm{Cyl}(A) \xrightarrow{p} A$$

of the fold map $id + id : A \amalg A \to A$, such that $i_0 + i_1$ is a cofibration and p is a weak equivalence.

(ii) A **left homotopy** from f to g consists of a cylinder object $A \amalg A \overset{i_0+i_1}{\to} \mathrm{Cyl}(A) \overset{p}{\to} A$ and a map $H : \mathrm{Cyl}(A) \to X$ such that $H(i_0 + i_1) = f + g$.



If there exists a left homotopy from f to g we say that f and g are *left homotopic* and write $f \overset{l}{\simeq} g$.

Lemma 2.4.1. *Suppose $\mathcal{C}$ is a model category. Every object A of $\mathcal{C}$ has a cylinder object $A \amalg A \overset{i_0+i_1}{\to} \mathrm{Cyl}(A) \overset{p}{\to} A$ in which p is a (necessarily acyclic) fibration.*

Proof. Apply the factorization axiom to the fold map $id + id : A \amalg A \to A$ to get $id + id = pi$ where p is an acyclic fibration and i is a cofibration. □

Dual to cylinder objects are *path objects* which generalize path spaces in the category of topological spaces.

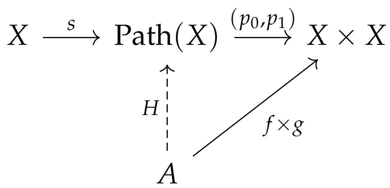
Definition 2.4.2. Suppose $\mathcal{C}$ is a model category and let $f, g : A \to X$ be morphisms in $\mathcal{C}$.

(i) A **path object** for X is an object $\mathrm{Path}(X)$ and a factorization

$$X \xrightarrow{s} \mathrm{Path}(X) \xrightarrow{(p_0,p_1)} X \times X$$

of the diagonal map $\Delta : X \to X \times X$, such that s is a weak equivalence and (p_0, p_1) is a fibration.

(ii) A **right homotopy** from f to g consists of a path object $X \overset{s}{\to} \mathrm{Path}(X) \overset{(p_0,p_1)}{\to} X \times X$ and a map $H : A \to \mathrm{Path}(X)$ such that $p_0 H = f$ and $p_1 H = g$.



If there exists a right homotopy from f to g we say that f and g are *right homotopic* and write $f \overset{r}{\simeq} g$.

Remark 2.4.1. There is no requirement that the choice of cylinder objects nor path objects be functorial.

Lemma 2.4.2. *Suppose $\mathcal{C}$ is a model category. Every object X of $\mathcal{C}$ has a path object $X \xrightarrow{s} \mathrm{Path}(X) \xrightarrow{(p_0,p_1)} X \times X$ in which s is a (necessarily acyclic) cofibration.*

Proof. Apply the factorization axiom to the diagonal map $\Delta : X \to X \times X$ to get a $\Delta = ps$ where p is an fibration and s is an acyclic cofibration. □

Lemma 2.4.1 and Lemma 2.4.2 show that, at least in this primitive sense, a model category has the structure needed to define homotopy relations.

Definition 2.4.3. Suppose $\mathcal{C}$ is a model category and let $f, g : A \to X$ be morphisms in $\mathcal{C}$.

(i) The morphisms f and g are **homotopic** if they are both left homotopic and right homotopic, this is written $f \simeq g$.
(ii) f is a **homotopy equivalence** if there is a map $h : X \to A$ such that $hf \simeq id_A$ and $fh \simeq X$.

Left and right homotopy relations

Left homotopy and right homotopy are relations on the set $\mathrm{Hom}_{\mathcal{C}}(A, X)$. They are not in general equivalence relations. Let $\pi^l(A, X)$ denote the set of equivalence classes of $\mathrm{Hom}_{\mathcal{C}}(A, X)$ under the equivalence relation *generated* by left homotopy. Dually, let $\pi^r(A, X)$ denote the set of equivalence classes of $\mathrm{Hom}_{\mathcal{C}}(A, X)$ under the relation *generated* by right homotopy.

Lemma 2.4.3. *If A is cofibrant and $A \amalg A \xrightarrow{i_0+i_1} \mathrm{Cyl}(A) \xrightarrow{p} A$ is a cylinder object for A, then $i_0, i_1 : A \to \mathrm{Cyl}(A)$ are acyclic cofibrations.*

Proof. Since $id : A \to A$ is a weak equivalence and since $A \xrightarrow{i_0} \mathrm{Cyl}(A) \xrightarrow{p} A$ is a factorization of id with p a weak equivalence, it follows from the 2-out-of-3 axiom that i_0 is a weak equivalence. The following diagram is a pushout in $\mathcal{C}$

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle in_0} \\ A & \xrightarrow{in_1} & A \amalg A \end{array}$$

Since $0 \hookrightarrow A$ is a cofibration (i.e. A is cofibrant) and since cofibrations are preserved under pushouts it follows that $in_0 : A \to A \amalg A$ is a cofibration. Thus $i_0 = (i_0 + i_1) \circ in_0$ is a cofibration. Switching the roles of i_0 and i_1 shows that i_1 is an acyclic cofibration too. □

Remark 2.4.2. The first part of the above proof does not use the cofibrancy hypothesis on A. Thus $i_0, i_1 : A \to \mathrm{Cyl}(A)$ are always weak equivalences.

Lemma 2.4.4. *Let $f, g : A \to X$ and suppose $f \overset{l}{\simeq} g$. Then f is a weak equivalence if and only if g is.*

Proof. By assumption there is some homotopy $H : \mathrm{Cyl}(A) \to X$ with $f = Hi_0$ and $g = Hi_1$. By Remark 2.4.2, i_0 and i_1 are weak equivalences. By the 2-out-of-3 axiom g is a weak equivalence if and only if H is a weak equivalence, which happens if and only if f is a weak equivalence. □

Lemma 2.4.5. *If A is cofibrant then left homotopy $\overset{l}{\simeq}$ is an equivalence relation on* $\mathrm{Hom}_{\mathcal{C}}(A, X)$

Proof. **Reflexivity** Let $f \in \mathrm{Hom}_{\mathcal{C}}(A, X)$. Choose any cylinder object $A \amalg A \overset{i_0+i_1}{\to} \mathrm{Cyl}(A) \overset{p}{\to} A$ and define H to be $f \circ p$.

Symmetry Let $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, X)$ and suppose $H : \mathrm{Cyl}(A) \to X$ is a left homotopy from some chosen cylinder object for A. The switch map $\tau : A \amalg A \to A \amalg A$ is easily seen to be a cofibration and thus we get a new cylinder object $A \amalg A \overset{\tau}{\to} A \amalg A \overset{i_0+i_1}{\to} \mathrm{Cyl}(A) \overset{p}{\to} A$. Now $H((i_0 + i_1)\tau) = g + f$ and so g is left homotopic to f.

Transitivity Suppose $f, g, h : A \to X$ are morphisms and that $(Cyl(A), H)$, $(\mathrm{Cyl}'(A), H')$ are left homotopies between f and g and between g and h, respectively. Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{i_0} & \mathrm{Cyl}(A) \\ \downarrow{\scriptstyle i_1'} & & \downarrow{\scriptstyle q} \\ \mathrm{Cyl}'(A) & \xrightarrow{q'} & C \end{array}$$

where, since A is assumed cofibrant, i_0 and i_1' are acyclic cofibrations. By Lemma 2.3.4, q and q' are acyclic cofibrations. Let $j = q \circ i_0$ denote the map $A \to C$. Thus j is an acyclic cofibration. The maps $p : \mathrm{Cyl}(A) \to A$ and $p' : \mathrm{Cyl}'(A) \to A$ define, via the universal property of the pushout square, a map $q'' : C \to A$. Since p and q are weak equivalences so is q'' (using the 2-out-of-3 property). Also, $q'' \circ j = id_A$. The map $j + j : A \amalg A \to C$ need not be a cofibration, but we can factor it (using the factorization axiom) as $A \amalg A \overset{j'}{\to} C' \overset{s}{\to} C$, a cofibration followed by an acyclic fibration. Then C' is a cylinder object for A, i.e. the map

$$A \amalg A \xrightarrow{j'} C' \xrightarrow{q'' \circ s} A$$

factors the fold map and $q'' \circ s$ is a weak equivalence and j' is a cofibration. Now the maps H and H' define, via the universal property, a map $H'' : C \to X$. Then $H'' \circ s$ defines a left homotopy from f to h.

□

Remark 2.4.3. We only used the hypothesis on A to show the transitivity property. So left homotopy is always reflexive and symmetric.

Lemma 2.4.6. *If X is fibrant then right homotopy $\overset{r}{\simeq}$ is an equivalence relation on* $\mathrm{Hom}_{\mathcal{C}}(A, X)$.

Proof. Dual to the proof of Lemma 2.4.5. □

Lemma 2.4.7. *Let $f, g : A \to X$ be morphisms.*

(i) If A is cofibrant and $f \overset{l}{\simeq} g$, then $f \overset{r}{\simeq} g$.
(ii) If X is fibrant and $f \overset{r}{\simeq} g$, then $f \overset{l}{\simeq} g$.

Proof. We show *(i)*, then *(ii)* follows by duality. Let $A \amalg A \overset{i_0+i_1}{\to} \mathrm{Cyl}(A) \overset{p}{\to} A$ and $H : \mathrm{Cyl}(A) \to X$ be the cylinder object and left homotopy that we assume exist. By Lemma 2.4.3, i_0 is an acyclic cofibration. Choose a path object

$$X \overset{s}{\to} \mathrm{Path}(X) \overset{(p_0,p_1)}{\to} X \times X$$

for X. Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{s \circ f} & \mathrm{Path}(X) \\ {\scriptstyle i_0}\downarrow & \overset{K}{\nearrow} & \downarrow{\scriptstyle (p_0,p_1)} \\ \mathrm{Cyl}(A) & \xrightarrow{(fp,H)} & X \times X. \end{array}$$

Since (p_0, p_1) is a fibration and i_0 is an acyclic cofibration, there is a lift $K : \mathrm{Cyl}(A) \to \mathrm{Path}(X)$ making the filled diagram commute. Now the map $Ki_1 : A \to \mathrm{Path}(X)$ is a right homotopy from f to g. □

Remark 2.4.4. In the proof of the lemma we chose an arbitrary path object for X and used the given left homotopy to create a right homotopy with the chosen path object as codomain. Thus, if $f, g : A \to X$ are maps where A is cofibrant and X is fibrant, then $f \simeq g$ if and only if $f \overset{l}{\simeq} g$ for a fixed cylinder object, if and only if $f \overset{r}{\simeq} g$ for a fixed path object.

Proposition 2.4.1. *If A and X are fibrant-cofibrant then the left-homotopy and right-homotopy relations on* $\mathrm{Hom}_{\mathcal{C}}(A, X)$ *coincide and are equivalence relations.*

Proof. Follows from Lemma 2.4.7. □

Lemma 2.4.8. *If A is cofibrant and $p : Y \to X$ is an acyclic fibration, then "composition with p" induces a natural bijection $p_* : \pi^l(A,Y) \to \pi^l(A,X)$ taking $[f]$ to $[pf]$.*

Proof. The map p_* is well defined since composition with p takes a given homotopy H to a homotopy pH between the images (by p_*) of Hi_0 and Hi_1.
To see that p_* is surjective, let $[f] \in \pi^l(A,X)$. Since A is cofibrant and p is an acyclic fibration there is a lift $g : A \to Y$ in the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & Y \\ \big\downarrow & \overset{g}{\nearrow} & \big\downarrow p \\ A & \xrightarrow{f} & X \end{array}$$

Now $pg = f$ and so $p_*([g]) = [f]$. For injectivity, suppose $f, g : A \to Y$ are left homotopic, say with homotopy $H : \mathrm{Cyl}(A) \to Y$. Now the diagram

$$\begin{array}{ccc} A \amalg A & \xrightarrow{f+g} & Y \\ \big\downarrow & \overset{\widehat{H}}{\nearrow} & \big\downarrow p \\ \mathrm{Cyl}(A) & \xrightarrow{H} & X \end{array}$$

has a lift $\widehat{H} : \mathrm{Cyl}(A) \to Y$ since $A \amalg A \to \mathrm{Cyl}(A)$ is a cofibration. Then $\widehat{H}$ is a left homotopy from f to g, i.e. $[f] = [g]$. □

Lemma 2.4.9. *If X is fibrant and $i : A \to B$ is an acyclic cofibration, then composition with i induces a natural bijection $i^* : \pi^r(B,X) \to \pi^r(A,X)$.*

Proof. Dual to Lemma 2.4.8. □

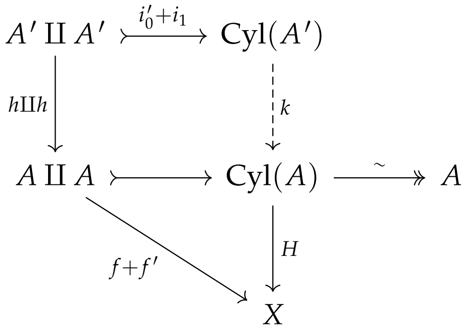
Lemma 2.4.10. *If X is a fibrant object, $f \overset{l}{\simeq} f' : A \to X$ and $h : A' \to A$ is a map, then $fh \overset{l}{\simeq} f'h$.*

Proof. We have a left homotopy $H : \mathrm{Cyl}(A) \to X$ from f to f'. Factor the map $\mathrm{Cyl}(A) \to A$ as $\mathrm{Cyl}(A) \to C \to A$ a (necessarily acyclic) cofibration followed by an acyclic fibration. Now let $\widehat{H} : C \to X$ be the lift in the diagram

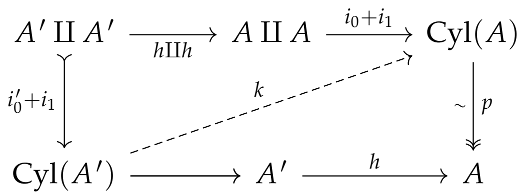
$$\begin{array}{ccc} \mathrm{Cyl}(A) & \xrightarrow{H} & X \\ \big\downarrow & \overset{\widehat{H}}{\nearrow} & \big\downarrow \\ C & \longrightarrow & 1 \end{array}$$

Then $\widehat{H}$ is also a homotopy from f to f'. Thus we may assume that $\mathrm{Cyl}(A) \to A$ is an acyclic fibration.

Choose a cylinder object $\mathrm{Cyl}(A')$ for A' and consider the diagram



where we wish to find k. We get k as a lift in the following diagram where we use the fact that $\mathrm{Cyl}(A) \to A$ is an acyclic fibration:



Now Hk is the desired left homotopy. □

Lemma 2.4.11. *If X is a fibrant object then using the composition law of $\mathcal{C}$ we get a well defined map*

$$\pi^l(A', A) \times \pi^l(A, X) \longrightarrow \pi^l(A', X) \qquad ([f], [g]) \longmapsto [gf]$$

Proof. Follows from Lemma 2.4.10 and the observation that if $f, f' : A' \to A$ are left homotopic via H, and $g : A \to X$ is some map, then gf is left homotopic to gf' via gH. □

Lemma 2.4.12. *If A is a cofibrant object then using the composition law of $\mathcal{C}$ we get a well defined map*

$$\pi^r(A, X) \times \pi^l(X, Y) \longrightarrow \pi^l(A, Y) \qquad ([f], [g]) \longmapsto [gf]$$

Proof. Dual to Lemma 2.4.11. □

Proposition 2.4.2. *Let $\mathcal{C}$ be a model category. Composition is well defined between homotopy classes of maps between fibrant-cofibrant objects of $\mathcal{C}$.*

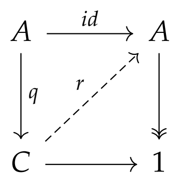
Proof. By Proposition 2.4.1 there is a well defined notion $\pi(A, X)$ of homotopy classes of maps between two fibrant-cofibrant objects A and X. Thus $\pi^l(A, X) = \pi^r(A, X) = \pi(A, X)$. The result now follows from Lemma 2.4.11 (or from Lemma 2.4.12). □

Whitehead's theorem

Whitehead's theorem from topology states that weakly homotopy equivalent maps between CW complexes are homotopy equivalences. The model category framework allows one to give a broad generalization of this theorem to arbitrary model categories.

Theorem 2.4.1 (Whitehead's Theorem)**.** *Suppose $\mathcal{C}$ is a model category. Suppose A and X are both fibrant and cofibrant in $\mathcal{C}$. Then $f : A \to X$ is a weak equivalence if and only if f is a homotopy equivalence.*

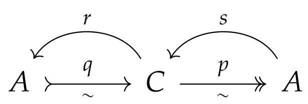
Proof. "$\Rightarrow$" Suppose $f : A \to X$ is a weak equivalence. Factor f as $A \overset{q}{\underset{\sim}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} X$ an acyclic cofibration followed by a fibration. Since f is a weak equivalence it follows that p is a weak equivalence. Since A is fibrant the diagram



may be filled, yielding a left inverse r to q. Since X is fibrant and q is an acyclic cofibration, the induced map $q^* : \pi^r(C, C) \to \pi^r(A, C)$ is a bijection by Lemma 2.4.9. Now

$$q^*([qr]) = [qrq] = [q] = q^*([id_C])$$

shows that $qr \overset{r}{\simeq} id_C$. Thus r is a two-sided right homotopy inverse to q. Dually, using that A is cofibrant and Lemma 2.4.8 one gets a two-sided left homotopy inverse $s : X \to C$ to p. Thus we have maps



and we claim that rs is a homotopy inverse to $f = pq$. This follows from the following calculations

$$f(rs) = p(qr)s \overset{r}{\simeq} ps = id_X$$

and

$$(rs)f = r(sp)q \overset{l}{\simeq} rq = id_A.$$

"$\Leftarrow$" Now suppose f has a homotopy inverse $g : X \to A$. Factor f as before, $A \overset{q}{\underset{\sim}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} X$. If we prove that p is a weak equivalence then we are done. Let $H : \mathrm{Cyl}(X) \to X$ be a left homotopy between fg and id_X. Since X is cofibrant

Lemma 2.4.3 implies that $i_0 : X \to \mathrm{Cyl}(X)$ is an acyclic cofibration. Thus there is a lift in the diagram

$$\begin{array}{ccc} X & \xrightarrow{qg} & C \\ {\scriptstyle \sim}\downarrow{\scriptstyle i_0} & \overset{H'}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathrm{Cyl}(X) & \xrightarrow{H} & X \end{array}$$

Let $s = H'i_1 : X \to C$. The map H' is a homotopy witnessing $s \simeq qg$. Then $ps = pH'i_1 = Hi_1 = id_X$. Let $r : C \to A$ be a homotopy inverse to q, which exists by "$\Rightarrow$" just proved. Then $fr = pqr \simeq p$. Thus we see

$$sp \simeq qgp \simeq qgfr \simeq qr \simeq id_C$$

where we have used Lemma 2.4.11 and the dual Lemma 2.4.12. Since id_C is a weak equivalence and $id_C \simeq sp$ it follows from Lemma 2.4.4 that sp is a weak equivalence. Now p is a retract of sp as seen by the commutativity of

$$\begin{array}{ccccc} C & \xrightarrow{id} & C & \xrightarrow{id} & C \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle sp} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{s} & C & \xrightarrow{p} & X \end{array}$$

Thus p is a weak equivalence. This completes the proof. □

Corollary 2.4.1. *Let $f, f' : A \to X$ be homotopic maps between fibrant-cofibrant objects and let $h : X \to Y$ be a weak equivalence, where Y is also fibrant-cofibrant. Then $hf \simeq hf'$ implies $f \simeq f'$.*

Proof. By Theorem 2.4.1, h has a homotopy inverse $h' : Y \to X$. Now $f \simeq h'hf \simeq h'hf' \simeq f'$ (compositions make sense since the objects are assumed fibrant-cofibrant), so $f \simeq f'$. □

The *classical* homotopy category

We can now describe the "classical homotopy category" of a model category $\mathcal{C}$. This is *not* the same as the "Quillen homotopy category associated to $\mathcal{C}$". In the classical homotopy category we include only the fibrant-cofibrant objects of $\mathcal{C}$. We have seen that the homotopy relation behaves very well between fibrant-cofibrant objects, reminiscent of the situation in topology.

Proposition 2.4.3. *Suppose $\mathcal{C}$ is a model category. There exists a category $\pi\mathcal{C}_{cf}$ whose objects are the fibrant-cofibrant objects of $\mathcal{C}$ and where morphisms between A and X from $\pi\mathcal{C}_{cf}$ is the set of equivalence classes $\pi(A, X)$.*

Proof. Follows from Proposition 2.4.2. □

Fibrant and cofibrant replacement

The next step towards the construction of the homotopy category $\mathrm{Ho}(\mathcal{C})$ associated to a model category $\mathcal{C}$ is that of fibrant and/or cofibrant replacement. This generalizes the CW approximations from topology and the projective/injective resolutions from homological algebra.
First we describe replacements of objects.

Definition 2.4.4. Suppose $\mathcal{C}$ is a model category and X is an object of $\mathcal{C}$.

(i) A **cofibrant replacement** to X is a weak equivalence $QX \xrightarrow{\sim} X$ where QX is a cofibrant object in $\mathcal{C}$.
(ii) A **fibrant replacement** to X is a weak equivalence $X \xrightarrow{\sim} RX$ where RX is a fibrant object in $\mathcal{C}$

We often refer simply to QX instead of the map $QX \xrightarrow{\sim} X$ when using the term *cofibrant replacement*. Similarly we use the term *fibrant replacement* to refer to RX instead of the map $X \xrightarrow{\sim} RX$.

Lemma 2.4.13. *Every object X in a model category has both a fibrant and a cofibrant replacement.*

Proof. Factor the unique map $0 \to X$ as a cofibration followed by an acyclic fibration, $0 \hookrightarrow QX \overset{\sim}{\twoheadrightarrow} X$. Likewise, factor the map $X \to 1$ as an acyclic cofibration followed by an fibration $X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow 1$. □

The replacements given by the factorization axiom (as in Lemma 2.4.13) have the extra property that the map $QX \to X$ is in fact a fibration, and that the map $X \to RX$ is in fact a cofibration. This motivates the next definition.

Definition 2.4.5. Suppose $\mathcal{C}$ is a model category and X is an object of $\mathcal{C}$.

(i) A **fibrant cofibrant replacement** to X is a cofibrant replacement $QX \overset{\sim}{\twoheadrightarrow} X$ which is a fibration.
(ii) A **cofibrant fibrant replacement** to X is a fibrant replacement $X \overset{\sim}{\hookrightarrow} RX$ which is a cofibration.

Of course an object X may have several replacements, so it is worthwhile to compare these.

Definition 2.4.6. Suppose $\mathcal{C}$ is a model category and X is an object of $\mathcal{C}$.

(i) A **map of cofibrant replacements** between two cofibrant replacements $Q_1X \xrightarrow{\sim} X$ and $Q_2X \xrightarrow{\sim} X$ is a map $Q_1X \to Q_2X$ in the over category $\mathcal{C}/X$.

(ii) A **map of fibrant replacements** between two fibrant replacements $X \underset{\sim}{\to} R_1X$ and $X \underset{\sim}{\to} R_2X$ is a map $R_1X \to R_2X$ in the under category $X/\mathcal{C}$.

In the familiar situations such as homological algebra, one proves equivalence up to homotopy of projective/injective resolutions. The corresponding statement in a general model category also holds.

Lemma 2.4.14. *Suppose $\mathcal{C}$ is a model category and X is an object of $\mathcal{C}$.*

(i) If $Q_1X \underset{\sim}{\twoheadrightarrow} X$ and $Q_2X \underset{\sim}{\twoheadrightarrow} X$ are two fibrant cofibrant replacements, then there is a map of cofibrant replacements $g : Q_1X \to Q_2X$ which is a homotopy equivalence

(ii) If $X \underset{\sim}{\hookrightarrow} R_1X$ and $X \underset{\sim}{\hookrightarrow} R_2X$ are two cofibrant fibrant replacements, then there is a map of fibrant replacements $g : R_1X \to R_2X$ which is a homotopy equivalence.

Proof. We prove *(i)*, *(ii)* is similar. Consider the commuting diagram

$$\begin{array}{ccc} 0 & \longrightarrow & Q_2X \\ \big\downarrow & \overset{g}{\nearrow} & \big\downarrow{\scriptstyle \sim} \\ Q_1X & \xrightarrow{\sim} & X \end{array}$$

The lift $g : Q_1X \to Q_2X$ exists since $Q_2X \underset{\sim}{\twoheadrightarrow} X$ is an acyclic cofibration, and since Q_1X is cofibrant. Now g is a weak equivalence by the 2-out-of-3 axiom. □

Remark 2.4.5. One can prove more precise uniqueness statements about replacements. For instance, one can show that arbitrary cofibrant replacements are "weakly equivalent" in the sense that there exists a "zig-zag" of weak equivalences (itself unique in a precise sense) connecting any two such replacements. For more details in this direction see [Hir03, chap. 8, and chap. 14].

We now describe replacements of maps in $\mathcal{C}$.

Definition 2.4.7. Let $\mathcal{C}$ be a model category and $f : X \to Y$ a map in $\mathcal{C}$.

(i) A **cofibrant replacement** of f consists of a cofibrant replacement QX of X, a cofibrant replacement QY of Y and a map $Qf : QX \to QY$ such that the following diagram commutes.

$$\begin{array}{ccc} QX & \xrightarrow{Qf} & QY \\ \big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle \sim} \\ X & \xrightarrow{f} & Y \end{array}$$

(ii) A **fibrant replacement** of f consists of a fibrant replacement RX of X, a fibrant replacement RY of Y and a map $Rf : RX \to RY$ such that the the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ RX & \xrightarrow{Rf} & RY \end{array}$$

Similarly one defines *fibrant cofibrant replacements* and *cofibrant fibrant replacements* of f by requiring that the replacements satisfy the obvious extra condition in each case.

Lemma 2.4.15. *Let $f : X \to Y$ be a map in the model category $\mathcal{C}$.*

(i) There is a fibrant cofibrant replacement Qf of f.
(ii) There is a cofibrant fibrant replacement Rf of f.

Proof. Factor $0 \to X$ as $0 \hookrightarrow QX \underset{\sim}{\twoheadrightarrow} X$. Now factor $QX \underset{\sim}{\twoheadrightarrow} X \xrightarrow{f} Y$ as a cofibration followed by an acyclic fibration $QX \hookrightarrow QY \underset{\sim}{\twoheadrightarrow} Y$. Then $QY \underset{\sim}{\twoheadrightarrow} Y$ is a cofibrant replacement of Y and $QX \hookrightarrow QY$ is a fibrant cofibrant replacement of f. The proof of *(ii)* is similar. □

Functorial replacements

So far we have not dealt with the possibility of getting *functorial* replacement functors $Q()$ and $R()$. There seems to be no reason why we should succeed in getting functorial replacement functors $\mathcal{C} \to \mathcal{C}$ since there may be many choices of replacements. But Lemma 2.4.14 tells us that the choices are never *homotopically* different (at least when we choose *fibrant* cofibrant replacements and *cofibrant* fibrant replacements), so perhaps we can get replacement functors $\mathcal{C} \to \pi\mathcal{C}_{cf}$ by passing to homotopy classes of maps.

Proposition 2.4.4. *Let $\mathcal{C}$ be a model category. There is a functor $\rho : \mathcal{C} \to \pi\mathcal{C}_{cf}$ which to every object X of $\mathcal{C}$ associates a fibrant replacement of a cofibrant replacement of X, and which to every map f associates the homotopy class of fibrant replacements of cofibrant replacements of f.*

Proof. We first choose specific fibrant replacements of cofibrant replacements for each object X of $\mathcal{C}$ (see Remark 2.4.7 below for some thoughts on the set theoretical issues here). *Choose* a factorization of $0 \to X$ as

$$0 \hookrightarrow QX \underset{\sim}{\overset{p_X}{\twoheadrightarrow}} X.$$

Now *choose* a factorization of $QX \to 1$ as

$$QX \overset{j_X}{\underset{\sim}{\hookrightarrow}} RQX \twoheadrightarrow 1.$$

If X is cofibrant to begin with then we choose $p_X = id_X$. If QX is fibrant then we choose $j_X = id_{QX}$. For all objects X the resulting object RQX is then fibrant and cofibrant. Define $\rho(X) = RQX$.
Given a map $f : X \to Y$, first lift f to $Qf : QX \to QY$ (where now QX and QY are fixed from the previous paragraph) as in the proof of Lemma 2.4.15. Now lift Qf to a map $RQf : RQX \to RQY$ (again RQX and RQY have already been chosen). Define $\rho(f) = [RQf] \in \pi(RQX, RQY)$ the homotopy class of RQf.
We check that ρ defines a functor. For $id : X \to X$ the map $id_{QX} : QX \to QX$ defines a cofibrant replacement of id_X with respect to the chosen replacement QX. The map $id_{RQX} : RQX \to RQX$ is a fibrant replacement of id_{QX}. Thus $[id_{RQX}] = \rho(id_X)$, and $[id_{RQX}]$ is the identity in $\pi\mathcal{C}_{cf}$.
Suppose we are given maps $f : X \to Y$ and $g : Y \to Z$. Consider the following diagram

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
\uparrow{\scriptstyle p_X} & & \uparrow{\scriptstyle p_Y} & & \uparrow{\scriptstyle p_Z} \\
QX & \xrightarrow{Qf} & QY & \xrightarrow{Qg} & QZ \\
\downarrow{\scriptstyle j_X} & & \downarrow{\scriptstyle j_Y} & & \downarrow{\scriptstyle j_Z} \\
RQX & \xrightarrow{RQ(f)} & RQY & \xrightarrow{RQ(g)} & RQZ
\end{array}$$

where each of the four small squares commute. By the commutativity of the top two squares we have $(p_Z)_*(Q(g) \circ Q(f)) = (p_Z)_*(Q(g) \circ Q(f))$. Since p_Z is an acyclic fibration, $(p_Z)_* : \pi^l(QX, QZ) \to \pi^l(QX, Z)$ is a bijection by Lemma 2.4.8. Thus $Q(g) \circ Q(f) \overset{l}{\simeq} Q(g \circ f)$.
Postcomposing with j_Z and using Lemma 2.4.11 we have

$$j_Z \circ Q(g) \circ Q(f) \overset{l}{\simeq} j_Z \circ Q(g \circ f).$$

By commutativity this means that

$$RQ(g) \circ RQ(f) \circ j_X \overset{l}{\simeq} RQ(g \circ f) \circ j_X.$$

Since QX is cofibrant Lemma 2.4.7 implies $RQ(g) \circ RQ(f) \circ j_X \overset{r}{\simeq} RQ(g \circ f) \circ j_X$. Finally by Lemma 2.4.9 we conclude $RQ(g) \circ RQ(f) \overset{r}{\simeq} RQ(g \circ f)$. Thus $\rho(g) \circ \rho(f) = \rho(g \circ f)$. $\square$

Remark 2.4.6. We could have defined a model category to have *functorial* factorizations. This would make the above proof much simpler. However, then we would also have had to check functoriality in each of the examples to come. In [Hir03] and [Hov99] functorial factorization is assumed while in [GJ99] and [DS95] it is not.

Remark 2.4.7. The proof of Proposition 2.4.4 requires a lot of choices! Since we have required our model categories to have all small limits, all (non trivial) examples will be *large* (see e.g. [Shu08]), i.e. the collection of objects will not form a set. So the usual axiom of choice will not suffice for the above proof. Of course if we assume as part of the model category structure some construction of the required factorizations, then this problem disappears. However we have avoided this assumption since it seems somewhat clumsy. Another possible solution would be to work inside the von Neumann-Bernays-Gödel (NBG) axiomatic system. The axiom of *global* choice is a consequence of the axioms of NBG (see [Shu08, chap. 7]) and this would therefore suffice for our purposes. One could also work inside a Grothendieck Universe where it is possible to be more delicate about size issues. We shall make no explicit choice of foundations in this thesis. While this state of affairs is not completely satisfactory, we will not delve deeper than this comment.

The Quillen homotopy category

We now construct Quillen's homotopy category $\mathrm{Ho}(\mathcal{C})$ associated to a model category $\mathcal{C}$. The objects of $\mathrm{Ho}(\mathcal{C})$ will be the same as those of $\mathcal{C}$ and we define

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) = \mathrm{Hom}_{\pi\mathcal{C}_{cf}}(\rho(X),\rho(Y)) = \pi(\rho(X),\rho(Y))$$

where $\rho : \mathcal{C} \to \pi\mathcal{C}_{cf}$ is the replacement functor defined in Proposition 2.4.4.

Remark 2.4.8. If X and Y are *fibrant-cofibrant* objects of $\mathcal{C}$ then the action of the functor ρ (defined in the proof of Proposition 2.4.4) is simply $\rho(X) = X$ and $\rho(Y) = Y$ and so $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) = \pi(X,Y)$.

There is a functor

$$\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$$

which is the identity on objects and takes a map f to $\rho(f)$.

If $f : X \to Y$ is a weak equivalence then $RQf : RQX \to RQY$ is a weak equivalence (by construction and using the 2-out-of-3 property twice). Since RQX and RQY are fibrant and cofibrant, Whitehead's theorem (Theorem 2.4.1) implies that $RQ(f)$ is a homotopy equivalence, i.e. $\gamma(f) = [RQf]$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. We will show that γ is universal amongst functors from $\mathcal{C}$ inverting the weak equivalences. First we need a lemma.

Lemma 2.4.16. *Let $\mathcal{C}$ be a model category and $\delta : \mathcal{C} \to \mathcal{D}$ be a functor which sends weak equivalences to isomorphisms. If $f, g : A \to X$ are maps such that either $f \overset{l}{\simeq} g$ or $f \overset{r}{\simeq} g$, then $\delta(f) = \delta(g)$.*

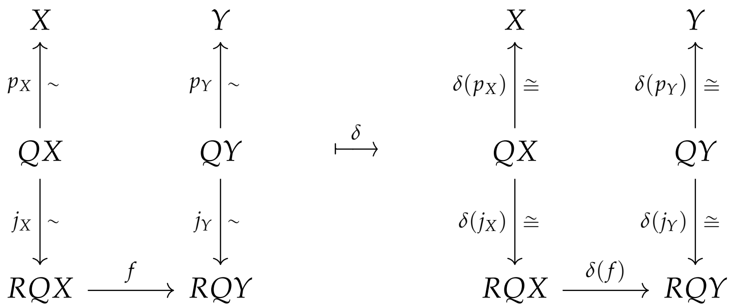
Proof. Suppose $f \overset{l}{\simeq} g$. Let $H : \mathrm{Cyl}(A) \to X$ be a left homotopy. By assumption $\delta(i_0)$ and $\delta(i_1)$ are isomorphisms (see Remark 2.4.2). Also, the map $\mathrm{Cyl}(A) \underset{\sim}{\overset{p}{\to}} A$ is a weak equivalence (so $\delta(p)$ is an isomorphism) and $pi_0 = pi_1 = id$. Thus $\delta(i_0) = \delta(i_1)$. Then

$$\delta(f) = \delta(H)\delta(i_0) = \delta(H)\delta(i_1) = \delta(g).$$

The proof in the case $f \overset{r}{\simeq} g$ is similar. □

Proposition 2.4.5. *The functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ is universal with respect to all functors from $\mathcal{C}$ which invert weak equivalences. More precisely, suppose $\delta : \mathcal{C} \to \mathcal{D}$ is a functor such that $\delta(f)$ is an isomorphism in $\mathcal{D}$ whenever f is a weak equivalence in $\mathcal{C}$. Then there is a unique functor $\delta_* : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ such that $\delta_* \circ \gamma = \delta$.*

Proof. We define δ_* to be δ on objects. Suppose $f : RQX \to RQY$ represents a morphism from X to Y in $\mathrm{Ho}(\mathcal{C})$. Then the diagrams



indicate that we may define

$$\delta_*([f]) = \delta(p_Y)\delta(j_Y)^{-1}\delta(f)\delta(j_X)\delta(p_X)^{-1},$$

which is well-defined by Lemma 2.4.16. This defines δ_* as a functor and we have $\delta_*\gamma = \delta$.

As for uniqueness, the maps p_X and j_X represent identity morphisms on RQX and so $[f] = \gamma(p_Y)\gamma(j_Y)^{-1}\gamma(f)\gamma(j_X)\gamma(p_X)^{-1}$ in $\mathrm{Ho}(\mathcal{C})$, so the value of δ_* on $[f]$ is determined. □

Remark 2.4.9. Proposition 2.4.5 is a way of stating that $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ is a *localization* of $\mathcal{C}$ with respect to the class $\mathcal{W}$ of weak equivalences. Such a localization is usually written $\mathcal{C} \to \mathcal{W}^{-1}\mathcal{C}$. It is a remarkable consequence of this that $\mathrm{Ho}(\mathcal{C})$, which we constructed using *all* the model structure (weak equivalences, fibration, and cofibrations), depends only on the class $\mathcal{W}$ of weak equivalences. A category $\mathcal{C}$ may well have many different model structures (for example the category of spaces has at least three different model structures), but if two model structures have the same weak equivalences then their homotopy categories will be equivalent.

Proposition 2.4.6. *The Quillen homotopy category* $\mathrm{Ho}(\mathcal{C})$ *is equivalent (as a category) to the classical homotopy category* $\pi\mathcal{C}_{cf}$*.*

Proof. It follows from Remark 2.4.8 that the inclusion $\varepsilon : \pi\mathcal{C}_{cf} \hookrightarrow \mathrm{Ho}(\mathcal{C})$ is a functor. Define $\eta : \mathrm{Ho}(\mathcal{C}) \to \pi\mathcal{C}_{cf}$ by $\eta(X) = \rho(X)$ and $\eta([f]) = [f]$. Then $\eta\varepsilon = id_{\pi\mathcal{C}_{cf}}$ and $\varepsilon \circ \eta \cong id_{\mathrm{Ho}(\mathcal{C})}$. □

Proposition 2.4.7. *Let* $\mathcal{C}$ *be a model category. A map* $f : X \to Y$ *in* $\mathcal{C}$ *is a weak equivalence if and only if* $\gamma(f)$ *is an isomorphism in* $\mathrm{Ho}(\mathcal{C})$*.*

Proof. It is clear that if f is a weak equivalence then $\gamma(f)$ is an isomorphism. Suppose $\gamma(f)$ is an isomorphism. Then $\gamma(f)$ is represented by a homotopy equivalence $\widetilde{f} : RQX \to RQY$. By Whitehead's Theorem (Theorem 2.4.1) $\widetilde{f}$ is a weak equivalence. Thus since the replacement morphisms j_X, j_Y, p_X, p_Y are all weak equivalences, it follows by the 2-out-of-3 axiom that f is a weak equivalence. □

2.5 Derived functors

Derived functors play an important role in this thesis. The total derived functor theorem is used several times to show that a pair of adjoint functors between model categories (subject to some constraints) induces an equivalence at the level of the homotopy categories.

Left and right derived functors

Suppose $\mathcal{C}$ is a model category and $F : \mathcal{C} \to \mathcal{D}$ is a functor with values in a category $\mathcal{D}$. It is natural to ask if F may be extended along $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ to yield a functor $\widetilde{F} : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ at the homotopy level. Thus we ask for a factorization

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle\gamma} & \nearrow_{\widetilde{F}} & \\ \mathrm{Ho}(\mathcal{C}) & & \end{array}$$

If F takes weak equivalences of $\mathcal{C}$ into isomorphisms in $\mathcal{D}$ then the universal property of $\mathrm{Ho}(\mathcal{C})$ (Proposition 2.4.5) shows that $\widetilde{F}$ exists. However, even when F does not invert weak equivalences some approximation to an extension along γ may exist. We may weaken the condition that the above diagram is commutative to require that there is simply a natural transformation between $\widetilde{F} \circ \gamma$ and F. This is made precise by the following definitions.

Definition 2.5.1. Let $\mathcal{C}$ be a model category, $\mathcal{D}$ a category, and $F : \mathcal{C} \to \mathcal{D}$ a functor.

(i) A **left derived functor** of F is a universal pair $(\mathbf{L}F, \varepsilon)$ of a functor $\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ and a natural transformation (*from the left*) $\varepsilon : \mathbf{L}F \circ \gamma \to F$.

Thus, given any pair (G, ζ) of a functor $G : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ and natural transformation $\zeta : G \circ \gamma \to F$ there must exist a unique natural transformation $\theta : G \to \mathbf{L}F$ such that $\zeta = \varepsilon(\theta_{\gamma(-)})$.

(ii) A **right derived functor** of F is a universal pair $(\mathbf{R}F, \eta)$ of a functor $\mathbf{R}F : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ and a natural transformation (*from the right*) $\eta : F \to \mathbf{R}F \circ \gamma$. Thus, given any pair (G, ζ) of a functor $G : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ and natural transformation $\zeta : F \to G \circ \gamma$ there must exist a unique natural transformation $\theta : \mathbf{R}F \to G$ such that $\zeta = (\theta_{\gamma(-)})\eta$.

Remark 2.5.1. As usual, the universality requirement implies uniqueness up to unique natural equivalence of left (or right) derived functors. Thus we speak of *the* left (or right) derived functor, if it exists.

Example 2.5.1. As hinted before the definition, if $F : \mathcal{C} \to \mathcal{D}$ inverts weak equivalences then the left (and right) derived functor $\mathbf{L}F$ (and $\mathbf{R}F$) exists. We may simply take $\mathbf{L}F = \widetilde{F}$ and $\varepsilon = id_F$ (likewise for $\mathbf{R}F$, thus $\mathbf{R}F = \mathbf{L}F$ in this case).

We will show that $F : \mathcal{C} \to \mathcal{D}$ has a left derived functor when F inverts weak equivalences between *cofibrant* objects of $\mathcal{C}$. In fact one only needs to check this for the acyclic cofibrations. Dually, to get a right derived functor it suffices to check the dual condition on the fibrant objects of $\mathcal{C}$. To make the theorem as sharp as possible we note the following result which will be useful later as well.

Lemma 2.5.1. *(Ken Brown's Lemma) Let $\mathcal{C}$ be a model category.*

(i) If $g : X \to Y$ is a weak equivalence between cofibrant objects of $\mathcal{C}$, then g may be factored as $g = ji$ where i is an acyclic cofibration, j is an acyclic fibration which has a right inverse which is an acyclic cofibration.

(ii) If $g : X \to Y$ is a weak equivalence between fibrant objects of $\mathcal{C}$, then g may be factored as $g = ji$ where i is an acyclic cofibration which has a left inverse which is an acyclic fibration, and j is an acyclic fibration.

Proof. Since X and Y are cofibrant the pushout square

$$\begin{array}{ccc} 0 & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \to & X \amalg Y \end{array}$$

shows that $X \to X \amalg Y$ and $Y \to X \amalg Y$ are cofibrations. Now factor the map $g + id : X \amalg Y \to Y$ as

$$X \amalg Y \overset{k}{\hookrightarrow} Z \underset{\sim}{\overset{j}{\twoheadrightarrow}} Y$$

where j is an acyclic fibration and k is a cofibration. The map $i : X \to X \amalg Y \to Z$ is a cofibration which is acyclic by the 2-out-of-3 axiom applied to g, j and i. Thus $X \underset{\sim}{\overset{i}{\hookrightarrow}} Z \underset{\sim}{\overset{j}{\twoheadrightarrow}} Y$ gives the factorization. Now the cofibration $r : Y \to X \amalg Y \to Z$ is a right inverse to j. The map r is acyclic by the 2-out-of-3 axiom. The proof of *(ii)* is similar. $\square$

Corollary 2.5.1. *Let $\mathcal{C}$ be a model category and $F : \mathcal{C} \to \mathcal{D}$ a functor.*

(i) If F takes acyclic cofibrations between cofibrant objects to isomorphisms, then F takes all weak equivalences between cofibrant objects to isomorphisms.
(ii) If F takes acyclic fibrations between fibrant objects to isomorphisms, then F takes all weak equivalences between fibrant objects to isomorphisms.

Proof. Suppose $f : X \to Y$ is a weak equivalence between cofibrant objects. By Lemma 2.5.1 we may factor $f = ji$ where i is an acyclic cofibration and j has a right inverse which is an acyclic cofibration. Apply F to the diagram $X \underset{\sim}{\hookrightarrow} Z \underset{\sim}{\twoheadrightarrow} Y$. By assumption $F(i)$ is an isomorphism and so is $F(r)$ (where r is the right inverse to j). Thus $F(j)$ is also an isomorphism, hence so is $F(f)$. Likewise for the dual statement. $\square$

Proposition 2.5.1. *Let $\mathcal{C}$ be a model category, $\mathcal{D}$ a category, and $F : \mathcal{C} \to \mathcal{D}$ a functor.*

(i) If F takes acyclic cofibrations between cofibrant objects to isomorphisms, then the left derived functor of F exists.
(ii) If F takes acyclic fibrations between fibrant objects to isomorphisms, then the right derived functor of F exists.

Proof. We show *(i)*. Let $Q(-)$ be a functorial fibrant cofibrant replacement. Q may be constructed as in the proof of Proposition 2.4.4 (see also Remark 2.4.7). We define $F_Q : \mathcal{C} \to \mathcal{D}$ by precomposition with Q, i.e. $F_Q(X) = F(Q(X))$ and $F_Q(f) = F(Q(f))$.
Now F_Q will invert weak equivalences. To see this, note that if $f : X \to Y$ is a weak equivalence then $Q(f)$ is a weak equivalence between cofibrant objects. By Corollary 2.5.1, $F_Q(f) = F(Q(f))$ is an isomorphism. Thus, by the universal property of $\mathrm{Ho}(\mathcal{C})$ (Proposition 2.4.5) there is an induced functor $\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ such that $\mathbf{L}F \circ \gamma = F_Q$. We need a natural transformation from $\mathbf{L}F \circ \gamma$ to F, but $\mathbf{L}F \circ \gamma = F_Q$ by construction. Thus we take $\varepsilon_X = F(QX \underset{\sim}{\twoheadrightarrow} X)$. We claim that $(\mathbf{L}F, \varepsilon)$ is in fact a left derived functor of F.
Suppose we are given (G, ζ), another left pair for F. Use the naturality of ζ with respect to the natural acyclic fibration $p_X : QX \underset{\sim}{\twoheadrightarrow} X$ to get the commuting square

$$\begin{array}{ccc} G\gamma(QX) & \xrightarrow{\zeta_{QX}} & F(QX) \\ \Big\downarrow{\scriptstyle G\gamma(p_X)} & & \Big\downarrow{\scriptstyle F(p_X)=\varepsilon_X} \\ G\gamma(X) & \xrightarrow{\zeta_X} & F(X) \end{array}$$

Since $\gamma(p_X)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ we may define

$$\theta_X : G \longrightarrow F(QX) = F_Q(X) = \mathbf{L}F \circ \gamma$$

by

$$\theta_X = \zeta_{QX} \circ (G\gamma(p_X))^{-1}.$$

Then θ_X is natural in X, i.e. defines a natural transformation $G \to \mathbf{L}F$ such that $\zeta_X = \varepsilon(\theta_{\gamma(-)})_X$. If X is cofibrant then $F(p_X) = \varepsilon_X$ is an isomorphism (as before by use of Corollary 2.5.1), thus the requirement $\zeta_X = \varepsilon(\theta_{\gamma(-)})_X$ determines θ_X. Since X is isomorphic to its cofibrant replacement in $\mathrm{Ho}(\mathcal{C})$ this shows that θ_X is unique (such that $\zeta_X = \varepsilon(\theta_{\gamma(-)})_X$) for all X. This completes the proof. □

Total derived functors

Definition 2.5.2. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and $F : \mathcal{C} \to \mathcal{D}$ a functor.

(i) A **total left derived functor** of F is a left derived functor of the composition $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\gamma_{\mathcal{D}}} \mathrm{Ho}(\mathcal{D})$ where γ_D is the localization functor for the model category $\mathcal{D}$. In more detail, a total left derived functor for F is a functor $\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ together with a natural transformation $\varepsilon : \mathbf{L}\gamma_{\mathcal{C}} \to \gamma_{\mathcal{D}} \circ F$ which is the "closest such pair from the left".

(ii) A **total right derived functor** of F is a right derived functor of the composition $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\gamma_{\mathcal{D}}} \mathrm{Ho}(\mathcal{D})$. In more detail, a total right derived functor for F is a functor $\mathbf{R}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ together with a natural transformation $\varepsilon : \gamma_{\mathcal{D}} \circ F \to \mathbf{R}F \circ \gamma_{\mathcal{C}}$ which is the "closest such pair from the right".

As a corollary to Proposition 2.5.1 we have the following proposition.

Proposition 2.5.2. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor between model categories.*

(i) If F takes acyclic cofibrations between cofibrant objects to weak equivalences in $\mathcal{D}$, then the total left derived functor of F exists.

(ii) If F takes acyclic fibrations between fibrant objects to weak equivalences in $\mathcal{D}$, then the total right derived functor of F exists.

Proof. By Proposition 2.4.7, $\gamma_{\mathcal{D}} \circ F$ satisfies the hypothesis of Proposition 2.5.1 and thus induces the left (respectively right) derived functors which are then seen to be total derived functors. □

2.6 Quillen adjunctions and Quillen equivalences

We now define the relevant notion of morphisms between model categories.

Definition 2.6.1. Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories.

(i) A functor $F : \mathcal{C} \to \mathcal{D}$ is called a **left Quillen functor** if F is a left adjoint and preserves cofibrations and acyclic cofibrations.
(ii) A functor $U : \mathcal{D} \to \mathcal{C}$ is called a **right Quillen functor** if U is a right adjoint and preserves fibrations and acyclic fibrations.
(iii) An adjunction $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ with F left adjoint to U is called a **Quillen adjunction** if F is a left Quillen functor.

Notation 2. Given an adjoint pair (F, U) with F left adjoint to U we follow Grothendieck's (and Quillen's) symphonic notation for the adjoint morphisms, i.e. given morphisms

$$f : FX \to Y \qquad \text{and} \qquad g : X \to UY$$

we let

$$f^\sharp : X \to UY \qquad \text{and} \qquad g^\flat : FX \to Y$$

denote the corresponding adjoint morphisms.

The definition of a Quillen adjunction seems asymmetric, however we could just as well define Quillen adjunctions by requiring U be a right Quillen functor. To see this we first need a lemma about adjunctions between Quillen categories.

Lemma 2.6.1. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories, (F, U) is an adjunction between them. Let $i \in \mathrm{Mor}(\mathcal{C})$ and $p \in \mathrm{Mor}(\mathcal{D})$. Then i has the LLP with respect to $U(p)$ if and only if p has the RLP with respect to $F(i)$. Thus, F preserves cofibrations if and only if U preserves acyclic fibrations. Likewise F preserves acyclic cofibrations if and only if U preserves fibrations.*

The proof uses the adjointness relations and is left to the reader.

Proposition 2.6.1. *An adjunction (F, U) between model categories is a Quillen adjunction if and only if U is a right Quillen functor.*

Proof. Follows directly from the Lemma 2.6.1. □

Lemma 2.6.2. *Let (F, U) be a Quillen pair. If A is cofibrant and $\mathrm{Cyl}(A)$ is a cylinder object for A, then $F(\mathrm{Cyl}(A))$ is a cylinder object for FA. Dually, if X is fibrant and $\mathrm{Path}(X)$ is a path object for X, then $U(\mathrm{Path}(X))$ is a path object for UX.*

Proof. Suppose A is cofibrant and $A \amalg A \hookrightarrow \mathrm{Cyl}(A) \underset{\sim}{\to} A$ is a cylinder object for A. Since F is left adjoint it preserves colimits so the natural map $F(A \amalg A) \overset{\cong}{\leftarrow} F(A) \amalg F(A)$ is an isomorphism. Thus $F(A \amalg A) \hookrightarrow F(\mathrm{Cyl}(A)) \to F(A)$ factors the fold map $id_{FA} + id_{FA}$ and $F(A \amalg A) \hookrightarrow F(\mathrm{Cyl}(A)$ is a cofibration since F is a left Quillen functor. Since A is cofibrant $i_0 : A \to \mathrm{Cyl}(A)$ is an acyclic cofibration (by Lemma 2.4.3) and so $F(i_0)$ is too. By the 2-out-of-3 axiom, $F(\mathrm{Cyl}(A)) \to FA$ is a weak equivalence, proving that $F(\mathrm{Cyl}(A))$ is a cylinder object for $F(A)$. The proof of the dual statement is similar. □

Total derived functor theorem

We can now state and prove a version of the *total derived functor theorem*.

Theorem 2.6.1. *(Total derived functor theorem) Let (F, U) be a Quillen pair between the model category $\mathcal{C}$ and $\mathcal{D}$. Then the total left derived functor $\mathbf{L}F$ of F exists, the total right derived functor $\mathbf{R}U$ of U exists and $\mathbf{L}F$ is left adjoint to $\mathbf{R}U$.*

Proof. By Proposition 2.5.2 both $\mathbf{L}F$ and $\mathbf{R}U$ exist. It remains to show that they are adjoint. Let $Q(-)$ be as in the proof of Proposition 2.5.1 and $R(-)$ the corresponding cofibrant fibrant replacement functor, let Q' and R' be the replacement functors for $\mathcal{D}$. The map $p_A : QA \underset{\sim}{\twoheadrightarrow} A$ becomes a bijection γp_A in $\mathrm{Ho}(\mathcal{C})$. Thus γp_A induces a bijection

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, \mathbf{R}UX) \xrightarrow[\cong]{(\gamma p_A)^*} \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(QA, \mathbf{R}UX) = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(QA, UR'X).$$

Now QA is cofibrant and $R'X$ is fibrant so Lemma 2.6.3 (proved below) gives a natural bijection

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(QA, UR'X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(FQA, R'X).$$

Working backwards we see $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(FQA, R'X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(\mathbf{L}FA, X)$ using the bijection induced by $\gamma j'_X$ (where $j'_X : X \underset{\sim}{\hookrightarrow} R'X$ is a weak equivalence). Thus $\mathbf{L}F$ is left adjoint to $\mathbf{R}U$. □

Lemma 2.6.3. *Let (F, U) be a Quillen pair between the model categories $\mathcal{C}$ and $\mathcal{D}$. Let $\mathbf{L}F$ and $\mathbf{R}U$ be the total derived functors. If A is cofibrant in $\mathcal{C}$ and X is fibrant in $\mathcal{D}$ then the adjoint isomorphism $\mathrm{Hom}_{\mathcal{D}}(FA, X) \cong \mathrm{Hom}_{\mathcal{C}}(A, \mathcal{U}X)$ induces a natural isomorphism $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(FA, X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, UX)$.*

Proof. By definition $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, UX) = \pi(RQA, RQUX)$. Since A is cofibrant $QA = A$ and $p_A = id_A$. Since X is fibrant UX is fibrant (since U preserves fibrations), thus $RQUX = QUX$ and $j_X = id_{QUX}$, since Q is a fibrant cofibrant replacement. Thus $\pi(RQA, RQUX) = \pi(RA, QUX)$. Now the map $j_A : A \underset{\sim}{\hookrightarrow} RA$ is an acyclic cofibration and so, since X is fibrant, Lemma 2.4.9 gives a natural isomorphism

$$\pi(RA, QUX) \cong \pi(A, QUX).$$

Likewise p_{QUX} is an acyclic fibration and so, since A is cofibrant, Lemma 2.4.8 gives a natural isomorphism

$$\pi(A, QUX) \cong \pi(A, UX).$$

Let $f, g : A \to UX$ be homotopic maps, say $H : \mathrm{Cyl}(A) \to UX$ is a homotopy. Then $H^\flat : F(\mathrm{Cyl}(A)) \to X$ is a homotopy between $f^\flat : FA \to X$ and $g^\flat : FA \to X$

by Lemma 2.6.2. Thus the natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(A, UX) \cong \mathrm{Hom}_{\mathcal{D}}(FA, X)$ induces a natural isomorphism $\pi(A, UX) \cong \pi(FA, X)$. Now working backwards we see that $\pi(FA, X)$ is naturally isomorphic to $\pi(RQFA, RQX)$ which establishes the claim. □

Quillen equivalences

Definition 2.6.2. Let (F, U) be a Quillen pair between model categories $\mathcal{C}$ and $\mathcal{D}$. Then F is called a **left Quillen equivalence**, U a **right Quillen equivalence** and (F, U) a **pair of Quillen equivalences** if the following condition is satisfied:

(i) For every cofibrant object A in $\mathcal{C}$ and fibrant object X in $\mathcal{D}$ and every map $g : A \to UX$ in $\mathcal{C}$, the map g is a weak equivalence in $\mathcal{C}$ if and only if the corresponding $g^\flat : FA \to X$ is a weak equivalence in $\mathcal{D}$.

Theorem 2.6.2. *(Total derived functor theorem for equivalences) Let (F, U) be a pair of Quillen equivalences. Then the induced derived functors $(\mathbf{L}F, \mathbf{R}U)$ form an adjoint equivalence of the categories $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{D})$.*

Proof. We have seen in Theorem 2.6.1 that the pair $(\mathbf{L}F, \mathbf{R}U)$ form an adjunction. It remains to show that, under the extra hypothesis, this adjunction is in fact an equivalence of categories. Let A be a cofibrant object of $\mathcal{C}$. The replacement map $FA \overset{j_{FA}}{\underset{\sim}{\hookrightarrow}} R'FA$ is a weak equivalence in $\mathcal{D}$. By assumption the adjoint map

$$j_{FA}^\sharp : A \xrightarrow[\sim]{} U(R'FA) = \mathbf{R}U(FA) = \mathbf{R}U(\mathbf{L}F(A))$$

is a weak equivalence. Here we have used that A is cofibrant to conclude $FA = \mathbf{L}FA$. Thus the induced map $\eta_A := \gamma(j_{FA}^\sharp)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. We have thus shown that the unit η_A of the adjunction is an isomorphism, at least for cofibrant A. But since any object of $\mathrm{Ho}(\mathcal{C})$ is isomorphic to an object which is cofibrant in $\mathcal{C}$ this shows that η is an natural isomorphism. Dually one shows that the counit is also a natural isomorphism. □

2.7 Two classical examples

As an illustration we provide the model structure on **kTop** and **sSet**. The proofs of the axioms are quite hard and will not be given here see [Qui67] or, for a more detailed approach [Hov99, chap. 2 and 3].

Spaces

We refer to objects of **kTop** simply as "topological spaces".

Definition 2.7.1. If $f : X \to Y$ is a continuous map of topological spaces then

(i) f is a *weak equivalence* if f induces a bijection of path components and an isomorphism of homotopy groups $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ for all $n \geq 1$ and all choices of basepoint $x_0 \in X$ (setting $y_0 = f(x_0)$).
(ii) f is a *fibration* if it is a Serre fibration (i.e. f has the RLP with respect to inclusions $D^n \hookrightarrow D^n \times I$ for all n).
(iii) f is a *cofibration* if it has the LLP with respect to all maps that are both fibrations and weak equivalences.

Theorem 2.7.1. *The category* **kTop** *with the proposed model structure of Definition 2.7.1 is a model category.*

Let $\mathbf{kTop}_*$ denote the category of *pointed* k-spaces with continuous maps preserving the base-point. Let $U : \mathbf{kTop}_* \to \mathbf{kTop}$ denote the forgetful functor.

Definition 2.7.2. If $f : X \to Y$ is a pointed continuous map of topological space then

(i) f is a *weak equivalence* if $U(f)$ is a weak equivalence in **kTop**.
(ii) f is a *fibration* if $U(f)$ is a fibration in **kTop**.
(iii) f is a *cofibration* if it is levelwise injective.

Proposition 2.7.1. *The category* $\mathbf{kTop}_*$ *with the proposed model structure of Definition 2.7.2 is a model category.*

Simplicial sets

Definition 2.7.3. If $f : X \to Y$ is a map of simplicial sets then

(i) f is a *weak equivalence* if the geometric realization $|f|$ of f is a weak equivalence of topological spaces.
(ii) f is a *fibration* if it is a Kan fibration (i.e. f has the right lifting property with respect to all horn-inclusions $\Lambda^n_k \to \Delta^n$ for $n > 0$ and $0 \leq k \leq n$).
(iii) f is a *cofibration* if it has the LLP with respect to all maps that are both fibrations and weak equivalences.

Theorem 2.7.2. *The category* **sSet** *with the proposed model structure of Definition 2.7.3 is a model category.*

Let $\mathbf{sSet}_*$ denote the category of *pointed* simplicial sets with base-point preserving simplicial maps. Let $U : \mathbf{sSet}_* \to \mathbf{sSet}$ be the forgetful functor.

Definition 2.7.4. If $f : X \to Y$ is a map of pointed simplicial sets then

(i) f is a *weak equivalence* if $U(f)$ is a weak equivalence in **sSet**.
(ii) f is a *fibration* if $U(f)$ is a fibration in **sSet**.
(iii) f is a *cofibration* if it has the LLP with respect to all maps that are both fibrations and weak equivalences.

Proposition 2.7.2. *The category* **sSet**$_*$ *with the proposed model structure of Definition 2.7.4 is a model category.*

Chapter 3

Simplicial Homotopy Theory

In this chapter we put a model structure on the category of r-reduced simplicial sets. We follow [Qui69, Part II, chap. 2].

3.1 r-Reduced simplicial sets

In this section we work in the category $\mathbf{sSet}_*$ of *pointed* simplicial sets.

Definition 3.1.1. A simplicial set K is called r-**reduced** if there is a unique k-simplex $x \in K$ for $0 \leq k \leq r$. In particular K is 1-reduced if K_0 and K_1 are both one-point sets. For $r \geq 0$ we let $\mathbf{sSet}_r$ denote the full subcategory of $\mathbf{sSet}_*$ consisting of r-reduced simplicial sets.

Note that an object in $\mathbf{sSet}_r$ automatically has a unique choice of basepoint.

Definition 3.1.2. For a simplical set K let $\mathbf{sk}_r K$ denote the r'th skeleton of K (i.e. the subcomplex generated by the simplices in dimension $\leq r$). Then $K/\mathbf{sk}_r K$ is clearly r-reduced. This defines an r-**reduction functor** $\mathbf{red}_r : \mathbf{sSet}_* \to \mathbf{sSet}_r$.

Definition 3.1.3. Given a simplicial set K (with basepoint x_0) let $E_r K$ denote the subcomplex of K consisting of those simplices $x \in K$ such that any face of x in dimension $\leq r$ is at the base point. This is the r'th **Eilenberg subcomplex**. More precisely, $x : \Delta^n \to K$ is in $E_r K$ if, whenever $t \leq r$ then the following diagram commutes

$$\begin{array}{ccc} \Delta^n & \xrightarrow{x} & K \\ \uparrow & & \uparrow{\scriptstyle x_0} \\ \Delta^t & \longrightarrow & * \end{array}$$

for any map $\Delta^t \to \Delta^n$.

The next proposition says that $\mathbf{sSet}_r$ is a reflective and coreflective subcategory of $\mathbf{sSet}_*$.

Proposition 3.1.1. *The inclusion functor $i : \mathbf{sSet}_r \to \mathbf{sSet}_*$ has a right and left adjoints. The left adjoint is given by the r-reduction functor $\mathbf{red}_r$. The right adjoint is given by the r'th Eilenberg subcomplex functor $E_r(-)$.*

Proof. If T is any r-reduced simplicial set and $f : K \to T$ is a simplicial map, then f induces a unique map $\widehat{f} : \mathbf{red}_r K \to T$ such that the following diagram commutes.

$$\begin{array}{ccc} K & \xrightarrow{f} & T \\ \downarrow & \nearrow_{\widehat{f}} & \\ \mathbf{red}_r K & & \end{array}$$

Thus $\mathbf{red}_r$ is left adjoint to the inclusion.
The right adjoint is given by the r'th Eilenberg subcomplex. If T is an r-reduced simplicial set and $f : T \to K$ is a pointed simplicial map then f clearly factors uniquely through the inclusion $E_r K \hookrightarrow K$, showing that $E_r(-)$ is right adjoint to the inclusion functor. □

Corollary 3.1.1. *The category $\mathbf{sSet}_r$ is both complete and cocomplete. Both limits and colimits are computed in $\mathbf{sSet}_*$.*

Proof. This follows from general category theory of reflective and coreflective subcategories of a complete and cocomplete category. □

Remark 3.1.1. If one worked without base points then colimits in $\mathbf{sSet}_r$ would not simply be computed in $\mathbf{sSet}_*$, we would have to compose with an Eilbenberg subcomplex, dependent on the choice of basepoint.

3.2 Model structure on $\mathbf{sSet}_r$

We will use the following simplicial version of a theorem of Serre.

Proposition 3.2.1. *Let $f : X \to Y$ be a morphism of 1-connected pointed simplicial sets. The following conditions are equivalent.*

(i) $\pi_* f \otimes \mathbb{Q} : \pi_* X \otimes \mathbb{Q} \xrightarrow{\cong} \pi_* Y \otimes \mathbb{Q}$ *is an isomorphism.*

(ii) $H_*(f;\mathbb{Q}) : H_*(X;\mathbb{Q}) \xrightarrow{\cong} H_*(Y;\mathbb{Q})$ *is an isomorphism.*

(iii) $f^* : H^*(Y, A) \xrightarrow{\cong} H^*(X, A)$ *is an isomorphism for all uniquely divisible Abelian groups A.*

See [Ber12, Theorem 4.1 and Lemma 4.8] for some wonderful proofs of these equivalences.

Definition 3.2.1. A map f which satisfies the equivalent conditions of Proposition 3.2.1 is called a *rational equivalence* or sometimes a $\mathbb{Q}$-*equivalence*.

Definition 3.2.2. Let f be a morphism in $\mathbf{sSet}_r$ for $r \geq 1$. We make the following definitions.

(i) Call f a **weak equivalence** if it is a $\mathbb{Q}$-equivalence (cf. Definition 3.2.1).
(ii) Call f a **cofibration** if it is injective.
(iii) Call f a **fibration** if it has the RLP with respect to all the acyclic cofibrations (i.e. the injective $\mathbb{Q}$-equivalence).

Remark 3.2.1. Note that a map f in $\mathbf{sSet}_r$ which is a Kan fibration in $\mathbf{sSet}_*$ is also a fibration in $\mathbf{sSet}_r$.

Remark 3.2.2. Note that the condition $r \geq 1$ is required to ensure that there is no fundamental group.

Theorem 3.2.1. *The category* $\mathbf{sSet}_r$ *(for* $r \geq 1$*) with the proposed model structure of Definition 3.2.2 is a model category, which we denote by* $\mathbf{sSet}_r^{\mathbb{Q}}$.

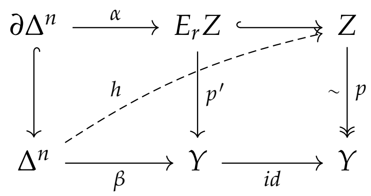
The proof of Theorem 3.2.1 is quite long, so we have spread it out as a series of lemmas. Note that Corollary 3.1.1 already shows that $\mathbf{sSet}_r$ is complete and cocomplete. The 2-out-of-3 property for the rational equivalences and the retract axioms are both readily verified. Also, by definition fibrations have the RLP with respect to acyclic cofibrations, which gives half of the lifting axiom.

Lemma 3.2.1. *Any map f in* $\mathbf{sSet}_r^{\mathbb{Q}}$ *may be factored as a cofibration followed by an acyclic fibration.*

Proof. Consider f as a map in $\mathbf{sSet}_*$. The model structure on $\mathbf{sSet}_*$ ensures the existence of a factorization $X \overset{i}{\hookrightarrow} Z \underset{\sim}{\overset{p}{\twoheadrightarrow}} Y$ where i is a cofibration and p is an acyclic Kan fibration. The simplicial set Z may not be r-reduced. However, taking the Eilenberg subcomplex E_rZ at the basepoint $i(x_0)$ (where x_0 is the basepoint of X) we get a commutative diagram

$$\begin{array}{ccccc} X & \overset{i}{\hookrightarrow} & Z & \underset{\sim}{\overset{p}{\twoheadrightarrow}} & Y \\ & {\scriptstyle i'}\searrow & \uparrow & \nearrow{\scriptstyle p'} & \\ & & E_rZ & & \end{array}$$

since X is r-reduced. Since i' is just i with a restricted codomain, it is injective. We must check that p' is an acyclic fibration. We will show that p' is infact an acyclic fibration in $\mathbf{sSet}_*$. Consider the diagram



The map $h : \Delta^n \to Z$ exists since p is an acyclic fibration in $\mathbf{sSet}_*$. We claim that h factors through E_rZ. We need only assume $n > r$ since if $n \leq r$ then the required lift $h' : \Delta^n \to E_rZ$ is given by the basepoint $z_0 : \Delta^n \to E_rZ$. Suppose $d : t \to n$ is some injection where $t \leq r$ in Δ and $x \in \Delta^n$. Then $d^*h(x) = h(d^*x)$ since h is a simplicial map. Then $d^*x \in \Delta^n_t$ but, $\Delta^n_t = (\partial\Delta^n)_t$ since $t \leq r < n$. Thus $h(d^*x) = \alpha(d^*x)$ is in E_rZ. So the image of h is contained in E_rZ.
This shows that p' is an acyclic Kan fibration, hence it is a fibration in $\mathbf{sSet}_r^{\mathbb{Q}}$ (cf. Remark 3.2.1). It is also a weak equivalence in $\mathbf{sSet}_r^{\mathbb{Q}}$ since it already induces isomorphisms on integral homotopy groups. □

Lemma 3.2.2. *Acyclic fibrations have the RLP with respect to cofibrations in* $\mathbf{sSet}_r^{\mathbb{Q}}$.

Proof. This is an application of the retract argument (Lemma 2.3.1). Let f be an acyclic fibration. Use Lemma 3.2.1 to factor $f = p'i'$ where p' is an acyclic fibration and i' is a cofibration. By the 2-out-of-3 property for weak equivalences, i' is a weak equivalence. Thus f (being a fibration) has the RLP with respect to i' and so is a retract of p'. Thus f is an acyclic fibration in $\mathbf{sSet}_*$, hence has the RLP with respect to injections, i.e. cofibrations. □

We also have the following corollary of the proof.

Corollary 3.2.1. *A map f in $\mathbf{sSet}_r$ is an acyclic fibration in $\mathbf{sSet}_r^{\mathbb{Q}}$ if and only if f is an acyclic Kan fibration in $\mathbf{sSet}_*$.*

It remains to prove the final factorization property in order to get a proof of Theorem 3.2.1. This will occupy us for quite a while.
First, we need a fact from homological algebra which will be useful. For an Abelian group A we let $A[n]$ be the chain complex concentrated in degree n where it has a copy of A. Similarly $A\langle n+1\rangle$ denotes the chain complex concentrated in degrees $n+1$ and n with a copy of A at these levels and with the identity map as the $n+1$st differential.

Lemma 3.2.3. *Let C_* be a chain complex and let $C^* = \mathrm{Hom}(C_*, A)$ be the associated cochain complex, with respect to the Abelian group A. Let Z^* be the cocycle's in C^*. Then there are natural isomorphisms $C^n \cong \mathrm{Hom}(C_*, A\langle n+1\rangle)$ and $Z^n \cong \mathrm{Hom}(C_*, A[n])$ (where the hom-sets are in the category of chain complexes).*

The proof is a straightforward unpacking of definitions.
In Section 8.4 we investigate the Dold-Kan correspondence, setting up an adjoint equivalence between simplicial Abelian groups and (non-negatively graded) chain complexes. In one direction, the functor

$$N^{-1} : \mathbf{dgAb} \to \mathbf{sAb}$$

(defined in Section 8.4) is left adjoint to the normalized chains functor $N : \mathbf{sAb} \to \mathbf{dgAb}$. By Moore's theorem that the homotopy groups of simplical Abelian groups may be computed as the homology of the normalized chain complex, one sees that $N^{-1}A[n]$ is an Eilenberg-Mac Lane object in $\mathbf{sAb}$, i.e.

$$\pi_k N^{-1}A[n] \cong H_n(NN^{-1}A[n]) \cong H_n(A[n]) = \begin{cases} A & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}.$$

We denote this object by $K(A, n)$, thereby fixing a specific model for the Eilenberg-Mac Lane object.

Proposition 3.2.2. *Let X be a simplicial set and A an Abelian group. Then there is a natural isomorphism $[X, K(A, n)] \cong H^n(X, A)$.*

See [GJ99, chap. 3, Theorem 2.19] for a proof.
If we apply N^{-1} to the short exact sequence

$$0 \to A[n] \to A\langle n+1\rangle \to A[n+1] \to 0$$

the result is a short exact sequence

$$0 \to K(A, n) \to WK(A, n) \xrightarrow{\varphi(A,n)} K(A, n+1) \to 0$$

where $WK(A, n)$ is contractible, by Whitehead's theorem (since $WK(A, n)$ is a Kan complex). We shall have great use of the map $\varphi(A, n)$ in a moment. The reason is the following. See Appendix A.1 for the notion of a minimal simplicial sets.

Lemma 3.2.4. *The Eilenberg-Mac Lane object $K(A, n)$ is a minimal Kan complex. It follows that the map $\varphi : WK(A, n) \to K(A, n+1)$ is a minimal fibration.*

See [GJ99, chap. 3, Lemma 2.21] for a proof.
An elementary, but important property of $\varphi(A, n)$ is that it represents the coboundary $\partial^* : C^n \to Z^{n+1}$. I.e. the coboundary ∂^* induces, under the natural isomorphisms of Lemma 3.2.3, the map $\varphi(A, n)$.
We will also need to know that the usual Postnikov tower construction can be carried out to yield a tower of *minimal* fibrations (see Appendix A.1 for the definition of a minimal fibration).

Proposition 3.2.3. *([Moo58, Proposition 2.18]) For a minimal Kan complex the Postnikov system consists of minimal fibrations.*

A proof of Proposition 3.2.3 is given in [May67, Lemma 12.1].
For a map f in $\mathbf{sSet}_*$ we let $\mathrm{Ker} f$ denote the fiber of f.

Lemma 3.2.5. *Let $f : Z \to X$ be a morphism in $\mathbf{sSet}_r$. If f is a (Kan) fibration in $\mathbf{sSet}_*$ and $\pi_*\mathrm{Ker} f$ is uniquely divisible, then f is a fibration in $\mathbf{sSet}_r^{\mathbb{Q}}$ such that $\pi_{r-1} f \otimes \mathbb{Q}$ is surjective.*

Proof. Consider the long exact sequence of homotopy groups, associated to the Kan fibration $f : Z \to Y$ in $\mathbf{sSet}_*$. The sequence looks like

$$\cdots \to \pi_{r-1} Z \otimes \mathbb{Q} \xrightarrow{\pi_{r-1} f \otimes \mathbb{Q}} \pi_{r-1} X \otimes \mathbb{Q} \xrightarrow{\partial} \pi_{r-2} \mathrm{Ker} f \otimes \mathbb{Q} \to \cdots$$

Since Z is r-reduced, $\mathrm{Ker} f \subseteq Z$ is also r-reduced, in particular is r-connected, hence $\pi_{r-2} \mathrm{Ker} f \otimes \mathbb{Q} = 0$, so $\pi_{r-1} f \otimes \mathbb{Q}$ is surjective.
It remains to show that f is a fibration in $\mathbf{sSet}_r^{\mathbb{Q}}$, i.e. has the RLP with respect to acyclic cofibrations. We will reduce the problem to showing that the maps $\varphi(A, n)$, constructed above, with $A = \pi_n \mathrm{Ker} f$, have the required lifting properties. To do this replace f by a minimal Kan fibration (see [May67, chap. 2, Theorem 10.9] for a general procedure which does this) i.e. factor $f = pq$ where q is an acyclic Kan fibration and $p : X \to Y$ is a minimal Kan fibration. Thus we are done if p has the required lifting property. Applying Proposition 3.2.3 we get a Postnikov system

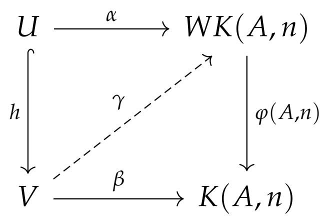
$$X = \varprojlim_n X_n \to \cdots \to X_n \xrightarrow{p_n} X_{n+1} \longrightarrow \cdots \longrightarrow X_{r-2} = Y$$

where each map p_n is a minimal fibration with fiber $K(A, n)$ where $A = \pi_n \mathrm{Ker}(p)$. Then $A \cong \pi_n \mathrm{Ker}(f)$ since the diagram,

$$\begin{array}{ccc} \mathrm{Ker}(f) & \longrightarrow & Z \\ \downarrow \sim & & \sim \downarrow q \\ \mathrm{Ker}(p) & \hookrightarrow & X \\ \downarrow & & \downarrow p \\ * & \longrightarrow & Y \end{array}$$

is a pullback in $\mathbf{sSet}_*$. Note that by assumption A is uniquely divisible. Proposition 3.2.2 above gives an isomorphism $H^{n+1}(Z, A) \cong [Z, K(A, n+1)]$. Furthermore, for Z a 1-connected set, the map $\varphi(A, n)$ induces a surjection from $[Z, K(A, n+1)]$ onto the set of isomorphism classes of minimal Kan fibrations with base Z and fiber $K(A, n+1)$ ([May67, Theorem 25.4]). A map $u : Z \to K(A, n+1)$ is mapped to the induced Kan fibration $u^*\varphi(A, n)$.
Since we assume $r \geq 1$ and each set X_n is r-reduced, they are also 1-connected. Thus the minimal fibrations p_n in the Postnikov tower are all induced as pullbacks

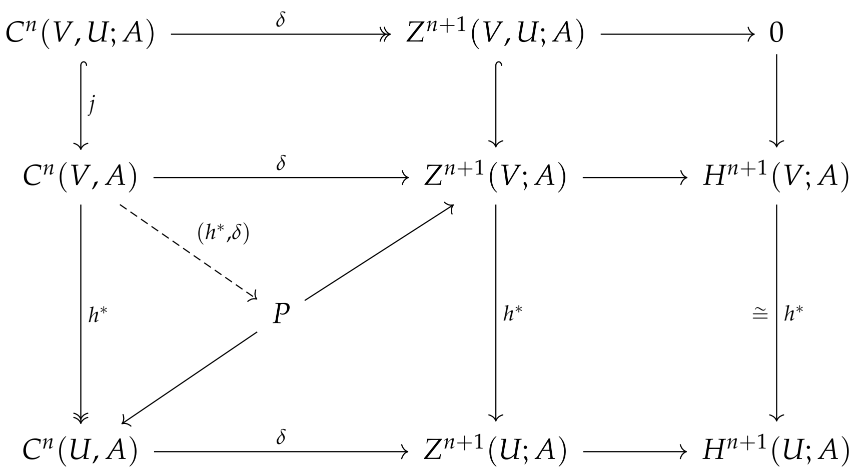
along $\varphi(A,n)$. Thus to show that p_n has the RLP, it suffices to show that $\varphi(A,n)$ has the RLP, with respect to acyclic cofibrations.
Let $h : U \underset{\sim}{\hookrightarrow} V$ be an acyclic cofibration, i.e. an injective $\mathbb{Q}$-equivalence of r-reduced simplicial sets. Let



be a commutative diagram representing a lifting problem. Finding the map γ making the diagram commute corresponds, by definition of $\varphi(A,n)$, to finding a normalized n-cochain $\tilde{\gamma} \in C^n(V;A)$ such that $h^* : C^*(V;A) \to C^*(U;A)$ maps $\tilde{\gamma}$ to $\tilde{\alpha}$ the normalized n-cochain $\tilde{\alpha} \in C^n(U;A)$ corresponding to α, and such that the coboundary δ maps $\tilde{\gamma}$ to the normalized n-cocycle $\tilde{\beta} \in Z^{n+1}$ corresponding to β. The pairs (α, β) making the above diagram commute are represented by elements of the pullback

$$\begin{array}{ccc} P & \longrightarrow & Z^{n+1}(V;A) \\ \downarrow & & \downarrow{\scriptstyle h^*} \\ C^n(U;A) & \xrightarrow{\delta} & Z^{n+1}(U;A) \end{array}$$

The induced map $h^* : C^*(V;A) \to C^*(U;A)$ is a surjective quasi-isomorphism of cochain complexes – surjectivity follows from the fact that A is an injective $\mathbb{Z}$-module. Suppose that the induced map $(h^*, \delta) : C^n(V;A) \to P$ is a surjection. Then there is some $\tilde{\gamma} \in C^n(V;A)$ solving the lifting problem. Thus we must show that $(h^*, \delta) : C^n(V;A) \to P$ is a surjection. Consider the following diagram



Given an element $(a, b) \in P$ we can lift a to an element $\tilde{a} \in C^n(V; A)$. Then $b - \delta(\tilde{a}) \in Z^{n+1}(V; A)$ is in the kernel of h^* and so is a relative (normalized) cocycle hence in $Z^{n+1}(V, U; A)$. Now the map $C^n(V, U; A) \to Z^{n+1}(V, U; A)$ is surjective since its cokernel is the kernel of the induced map h^* on cohomology. This kernel is trivial since by assumption h^* induces an isomorphism on cohomology with coefficients in A since A is uniquely divisible (cf. Definition 3.2.1). We therefore get a relative chain $\tilde{c} \in C^n(V, U; A)$. One can now check that $\tilde{a} - j(\tilde{c}) \in C^n(V; A)$ is mapped to (a, b) by (h^*, δ). This completes the proof. □

In fact the converse of Lemma 3.2.5 also holds, giving us a good characterization of those fibrations in $\mathbf{sSet}_r^{\mathbb{Q}}$ whose induced map on $\pi_{r-1}(-) \otimes \mathbb{Q}$ is surjective.

Lemma 3.2.6. *Let $f : Z \to Y$ be a map in $\mathbf{sSet}_r$ such that $\pi_{r-1} f \otimes \mathbb{Q}$ is surjective. Then $f = pi$ where i is an acyclic cofibration in $\mathbf{sSet}_r^{\mathbb{Q}}$ and where p is a Kan fibration such that $\pi_* \mathrm{Ker} p$ is uniquely divisible.*

We will not prove this lemma. See [Qui69, part II, Lemma 2.5] for a proof.
Putting the two lemma's together we have the following corollary.

Corollary 3.2.2. *The following conditions are equivalent for a map f in $\mathbf{sSet}_r$.*

(i) The map f is a fibration in $\mathbf{sSet}_r^{\mathbb{Q}}$ with $\pi_{r-1} f \otimes \mathbb{Q}$ surjective.
(ii) The map f is a Kan fibration in $\mathbf{sSet}_*$ and $\pi_* \mathrm{Ker} f$ is uniquely divisible.

Proof. The implication $(ii) \Rightarrow (i)$ is Lemma 3.2.5. For $(i) \Rightarrow (ii)$ factor $f = pi$ as guaranteed by Lemma 3.2.7. Now the retract argument applies since i has the LLP with respect to f (since f is assumed a fibration). Thus f is a retract of p and so satisfies (ii). □

Corollary 3.2.3. *The fibrant objects of $\mathbf{sSet}_r^{\mathbb{Q}}$ are the r-reduced Kan complexes with uniquely divisible homotopy groups.*

We are not quite done with showing the final factorization axiom for $\mathbf{sSet}_r^{\mathbb{Q}}$.

Lemma 3.2.7. *If f is a map in $\mathbf{sSet}_r$ then f may be factored as jg where j is an injective fibration in $\mathbf{sSet}_r^{\mathbb{Q}}$ and $\pi_{r-1} g \otimes \mathbb{Q}$ is surjective.*

See [Qui69, Part II, Lemma 2.7] for a proof of this lemma.
Putting these facts together we get the final factorization axiom.

Proposition 3.2.4. *Any map f in $\mathbf{sSet}_r$ may be factored as an acyclic cofibration in $\mathbf{sSet}_r^{\mathbb{Q}}$ followed by a fibration in $\mathbf{sSet}_r^{\mathbb{Q}}$.*

Proof. Factor f as jg as in Lemma 3.2.7 such that $\pi_{r-1} g \otimes \mathbb{Q}$ is surjective, and j is an injective fibration in $\mathbf{sSet}_r^{\mathbb{Q}}$. By Lemma 3.2.7 we may further factor g as pi where i is an acyclic cofibration in $\mathbf{sSet}_r^{\mathbb{Q}}$ and p is a Kan fibration such that $\pi_* \mathrm{Ker} p$ is uniquely divisible. Now $f = (jp)i$ and this is indeed a factorization of the required form. □

This completes the proof of the model category axioms for $\mathbf{sSet}_r^Q$, i.e. the proof of Theorem 3.2.1.

3.3 Representable simplicial sets

The category of simplicial sets $\mathbf{sSet} = \mathrm{Set}^{\Delta^{op}}$ is a presheaf category. The Yoneda embedding $y : \Delta \to \mathbf{sSet}$ associates to each ordinal n the *representable* functor $\mathrm{Hom}_\Delta(-, n) : \Delta^{op} \to \mathrm{Set}$, i.e. a simplicial object. Mostly we denote $\mathrm{Hom}_\Delta(-, n)$ by Δ^n, alluding to the fact that the geometric realization (defined below) of this simplicial set is (homeomorphic to) the standard n-simplex in $\mathbb{R}^{n+1}$. As with any presheaf category, arbitrary objects are colimits of representable objects. This works as follows. Given a simplicial set X, form the comma category $y \downarrow X$. The objects are pairs $(m, \mu : \Delta^m \to X)$ where m is an object of Δ and μ is a map of simplicial sets. The morphisms in $y \downarrow X$ between (m, μ) and (m', μ') are morphisms $f : m \to m'$ in Δ such that the diagram

$$\begin{array}{ccc} \Delta^m & \xrightarrow{f_*} & \Delta^{m'} \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu'} & \\ & X & \end{array}$$

commutes. Using the Yoneda lemma this can be described as follows. The objects in $y \downarrow X$ are pairs (m, x) where m is some finite ordinal and $x \in X_m$. The maps between (m, x) and (m', x') are the maps $f : m \to m'$ such that $X(f)(x') = x$.

"Forgetting" the second coordinates in $y \downarrow X$ gives a functor $U_X : y \downarrow X \to \Delta$ which takes an object (m, μ) to μ. Composing with the Yoneda embedding gives a diagram

$$y \downarrow X \xrightarrow{U_X} \Delta \xrightarrow{y} \mathbf{sSet}$$

of simplicial sets. "Forgetting" the first coordinates gives a natural transformation $\rho : y \circ U \to \delta_X$ where δ_X is the constant diagram $y \downarrow X \to \mathbf{sSet}$ with value X. The component of ρ at (m, μ) is simply $\mu : \Delta^m \to X$. The pair (X, ρ) is a cocone over $y \circ U_X$

Proposition 3.3.1. *The pair (X, ρ) is a colimit for the diagram $y \circ U_X$.*

The proof follows from the Yoneda lemma, [Mac71, chap. 3, sec. 7]. We will follow a common abuse of notation and write $X = \mathrm{colim}(y \circ U_X)$, suppressing the natural transformation ρ.

3.4 The geometric realization

Recall the standard **topological n-simplex** $|\Delta^n|$, defined as the subset of $\mathbb{R}^{n+1}$ given by

$$|\Delta^n| = \{ \sum_{i=0}^{n} t_i e_i \mid t_i \geq 0 \text{ and } \sum t_i = 1\}$$

where $(e_0, \ldots, e_i, \ldots, e_n)$ are the standard basis vectors in $\mathbb{R}^{n+1}$. We can use Proposition 3.3.1 to construct the geometric realization functor $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$. In fact the definition is forced on us if we assume that the realization of $y(n) = \Delta^n$ is the topological n-simplex and that $|\cdot|$ preserves colimits.

Definition 3.4.1. Let X be a simplicial set. The **geometric realization** of X is the space $\operatorname{colim}_{y\downarrow X}|\Delta^n|$.

The diagram $y \downarrow X \to \mathbf{Top}$ is the same as before, only this time $y : \Delta \to \mathbf{Top}$ associates to each finite ordinal the space $|\Delta^n|$. For a map $f : m \to n$ the induced map $y(f) = f_* : |\Delta^n| \to |\Delta^m|$ is the map defined by

$$f\left(\sum t_i e_i\right) = \sum t_i e_{f(i)}.$$

Note that since a map of simplicial sets $f : X \to Y$ is a natural transformation, it induces a continuous map on the colimits $|f| : |X| \to |Y|$ which respects composition and identites. Thus $|\cdot|$ is a functor $\mathbf{sSet} \to \mathbf{Top}$.
A more concrete description of $|X|$ is as follows. Consider the space

$$\coprod_{n\in\omega} X_n \times |\Delta^n|$$

Now let $\sim$ be the equivalence relation generated by the following relation. For $\theta : m \to n$ a morphism in Δ we require

$$(\theta^*(x), t) \sim (x, \theta_*(t))$$

for all $x \in X_n$ and all $t = \sum t_i e_i \in |\Delta^m|$. Then

$$|X| = \left(\coprod_{n\in\omega} X_n \times |\Delta^n|\right) / \sim .$$

3.5 The singular simplicial set

If X is a topological space then a **singular n-simplex** in X is a continuous map $\sigma : |\Delta^n| \to X$. Given a map $\theta : m \to n$ in Δ and a singular n-simplex $\sigma : |\Delta^n| \to X$ the map $\sigma \circ \theta_* : |\Delta^m| \to X$ is a singular m-simplex. This defines a simplicial set $\operatorname{Sing}(X)$ called the **singular simplicial set** of X. A continuous map $f : X \to Y$ defines a natural transformation $\operatorname{Sing}(X) \to \operatorname{Sing}(Y)$ taking $\sigma : |\Delta^n| \to X$ to $f \circ \sigma : |\Delta^n| \to Y$. This defines a functor $\operatorname{Sing}(-) : \mathbf{Top} \to \mathbf{sSet}$.

Proposition 3.5.1. *The geometric realization is left adjoint to the singular simplicial set.*

Proof. The proof works by first proving the statement for the representable objects, i.e. $\mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{Sing}(Y))$. This is a bijection is almost by definition, since a continuous map $\sigma : |\Delta^n| \to Y$ is an element $\sigma \in \mathrm{Sing}(Y)_n$, so by the Yoneda lemma it corresponds naturally to a unique map of simplicial sets $\Delta^n \to \mathrm{Sing}(Y)$.
We now have the following sequence of natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Top}}(|X|, Y) &\cong \lim_{y \downarrow X} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) \\ &\cong \lim_{y \downarrow X} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{Sing}(Y)) \\ &\cong \mathrm{Hom}_{\mathbf{sSet}}(X, \mathrm{Sing}(Y)) \end{aligned}$$

using that $\mathrm{colim}_{y \downarrow X} \Delta^n \cong X$ and that representable functors map colimits to limits. □

Proposition 3.5.2. *The geometric realization $|X|$ of a simplicial set is a CW-complex.*

3.6 Quillen equivalence

As a result of Proposition 3.5.2 we can view the geometric realization as a functor $|\cdot| : \mathbf{sSet} \to \mathbf{kTop}$ taking values in the category of k-spaces (see Appendix A.2). The category k-spaces admits a model structure (see [Hov99, chap. 2, Theorem 2.4.23]) induced by the model structure on **Top**.

Theorem 3.6.1. *The geometric realization and singular simplicial set form a Quillen equivalence between* **kTop** *and* **sSet**. *They also form Quillen equivalences between the corresponding pointed model categories* $\mathbf{kTop}_*$ *and* $\mathbf{sSet}_*$.

The proof of this theorem is long and hard. A relatively detailed exposition may be found in [Hov99, chap. 3, sec. 6] (See also [GZ67, chap. 7]).
The Quillen equivalence just exhibited above can *not* be refined to yield a Quillen equivalence between $\mathbf{Top}_r$ ($r \geq 1$) (the category of pointed r-connected topological spaces) and $\mathbf{sSet}_r^{\mathbb{Q}}$ for the trivial reason that $\mathbf{Top}_r$ does not admit a model structure. But $\mathbf{Top}_r$ still has a useful notion of homotopy, namely the usual one! Thus we can get a result of the same flavor as a Quillen equivalence.

Definition 3.6.1. Define a map $f : X \to Y$ in $\mathbf{Top}_r$ to be a *weak equivalence* if it is a *rational equivalence*, i.e. if $\pi_n(f) \otimes \mathbb{Q} : \pi_n(X, x_0) \otimes \mathbb{Q} \longrightarrow \pi_n(Y, y_0) \otimes \mathbb{Q}$ is an isomorphism.

Since $\mathbf{Top}_r$ is not a model category our construction of the associated homotopy category (Section 2.4) can not be used. Instead we make an "external" localization with respect to the weak equivalences. Let $\mathcal{W}$ denote the class of weak equivalences in $\mathbf{Top}_r$ and let $\gamma : \mathbf{Top}_r \to \mathcal{W}^{-1}\mathbf{Top}_r$ be a localization of $\mathbf{Top}_r$ with respect to the maps in $\mathcal{W}$.

Theorem 3.6.2. *The functors* $|\cdot| : \mathbf{sSet}_r^Q \to \mathbf{Top}_r$ *and* $E_r\mathrm{Sing} : \mathbf{Top}_r \to \mathbf{sSet}_r^Q$ *both preserve weak equivalences. Furthermore the counit* $\varepsilon_X : |E_r\mathrm{Sing}(X)| \to X$ *is a weak equivalence for all spaces* X*. As a consequence* $|\cdot|$ *and* $E_r\mathrm{Sing}$ *induce an equivalence of the localized categories* $\mathcal{W}^{-1}\mathbf{Top}_r$ *and* $\mathrm{Ho}(\mathbf{sSet}_r^Q)$.

Proof. The geometric realization preserves weak equivalences by Definition 3.2.2. If K is an r-connected pointed Kan complex (e.g. if K is r-reduced) then the r-th Eilenberg subcomplex E_rX is also a Kan complex, and the inclusion $E_rK \to K$ is a weak equivalence in **sSet** [Moo58, Proposition 2.7]. Suppose $f : X \to Y$ is a weak equivalence in $\mathbf{Top}_r^Q$. Then

$$\pi_n E_r \mathrm{Sing} X \xrightarrow{f_*} \pi_n E_r \mathrm{Sing} Y$$

is an isomorphism since $\pi_n K \cong \pi_n |K|$ for a Kan complex.
From Theorem 3.6.1 we know that the unit $K \to \mathrm{Sing}|K|$ is a weak equivalence in **sSet**. Recall that $\mathrm{Sing}X$ is a Kan complex for any space X and that homotopy groups of Kan complexes may be computed directly in **sSet**. If X is r-connected then

$$\pi_* X \cong \pi_* \mathrm{Sing} X \cong \pi_* E_r \mathrm{Sing} X \cong \pi_* |E_r \mathrm{Sing} X|$$

and so ε is a weak equivalence. $\square$

Chapter 4

Simplicial Groups

In this chapter we introduce the two functors G and $\overline{W}$ and to prove that they are adjoint. We then consider the problem of putting a model structure on the category of r-reduced simplicial groups. Finally we sketch an argument showing that G and $\overline{W}$ form a Quillen equivalence.

4.1 Kan's loop group

Let $\mathbf{sSet}_0$ be the category of reduced simplicial sets, and $\mathbf{sGrp}$ the category of simplicial groups. We define functors

$$\mathbf{sSet}_0 \underset{\overline{W}}{\overset{G}{\rightleftarrows}} \mathbf{sGrp}$$

and prove various properties about them, chief among these being that G and $\overline{W}$ induce equivalences after localization with respect to weak homotopy.

Definition 4.1.1. Let K be a reduced simplicial set. The **loop group** of K, denoted GK is the simplicial group with

$$GK_n = F(K_{n+1})/F(s_0K_n)$$

where $F(X)$ denotes the free group on X. Note that GK_n is isomorphic to $F(K_{n+1} - s_0K_n)$ the free group on $K_{n-1} - s_0K_n$, in particular each group GK_n is free. We define the face and degeneracy maps by the equations

$$\begin{aligned} d_0[x] &= [d_1x]\cdot[d_0x]^{-1} & \\ d_i[x] &= [d_{i+1}x] & i > 0 \\ s_i[x] &= [s_{i+1}x] & i \geq 0 \end{aligned}$$

It is straight forward to check that these definitions are well-defined and satisfy the simplicial identities. Furthermore given a map of reduced simplicial sets $f : K \to K'$ we get a map $Gf : GK \to GK'$ given level-wise as $Gf_n[x] = [f_{n+1}x]$. This defines the functor $G(-)$ from $\mathbf{sSet}_0$ to $\mathbf{sGrp}$.

Theorem 4.1.1. *(Moore) The underlying simplicial set of any simplicial group is a Kan complex.*

The proof of this theorem is reproduced in several places, e.g. [GJ99, chap. 1, Lemma 3.4] and [May67, Theorem 17.1], however they are often very brief. See [GM03, chap. 5, sec. 2] for detailed proof.
In particular Kan's loop group GK is a Kan complex.

Definition 4.1.2. Let G be a simplicial group. We define the **simplicial classifying space** for G, denoted $\overline{W}G$, to be the reduced simplicial set given by

$$\overline{W}G_n = G_{n-1} \times G_{n-2} \times \cdots \times G_0, \qquad n > 0$$

and with $\overline{W}G_0 = *$. The simplicial operators are given by

$$\begin{aligned} d_0(g_{n-1}, \ldots, g_0) &= (g_{n-2}, \ldots, g_0) \\ d_i(g_{n-1}, \ldots, g_0) &= (d_{i-1}g_{n-1}, d_{i-2}g_{n-2}, \ldots, d_0 g_{n-i} \cdot g_{n-i-1}, g_{n-i-2}, \ldots, g_0) & i > 0 \\ s_i(g_{n-1}, \ldots, g_0) &= (s_{i-1}g_{n-1}, s_{i-2}g_{n-2}, \ldots, s_0 g_{n-i}, e_{n-i}, g_{n-i-1}, \ldots, g_0) & i \geq 0 \\ s_0(g_{n-1}, \ldots, g_0) &= (e_n, g_{n-1}, \ldots, g_0) \end{aligned}$$

where e_j is the unit in G_j.

Proposition 4.1.1. *$\overline{W}G$ is a Kan complex.*

See [GJ99, Corollary 6.8].

4.2 Adjointness

Let $f : GK \to H$ be a map of simplicial groups and let $\alpha(f) : K \to \overline{W}H$ be defined by

$$\alpha(f)(x) = (f[x], f[d_0 x], f[d_0^2 x], \cdots, f[d_0^{n-1} x])$$

for $x \in K_n$. Conversely, given $g : K \to \overline{W}H$ a map of reduced simplicial sets, define $\beta(g) : GK \to H$ by

$$\beta(g)[x] = y_{n-1}$$

where y_{n-1} is the first entry in the n-tuple $g(x) = (y_{n-1}, y_{n-2}, \cdots, y_0) \in \overline{W}H$. It is not hard to verify that α and β witness that G is left adjoint to $\overline{W}$.

Proposition 4.2.1. *The functor G is left adjoint to $\overline{W}$. In particular, the maps α and β are natural inverse bijections between $\mathrm{Hom}_{\mathbf{sSet}_0}(K, \overline{W}H)$ and $\mathrm{Hom}_{\mathbf{sGrp}}(GK, H)$.*

See also [Kan58, Proposition 10.5].

4.3 GK is a loop group

In the above construction of GK we called it the *loop group* for K. We briefly justify using this term.

Definition 4.3.1. A **principal bundle** is a triple (G, t, K) where G is a simplicial group, K is a reduced simplicial set, and $t : K \to G$ is a function (called a **twisting function**) of degree -1 satisfying

$$\begin{aligned} d_i t(x) &= t(d_{i+1}x) & i > 0 \\ d_0 t(x) &= t(d_1 x) \cdot t(d_0 x)^{-1} & \\ s_i t(x) &= t(s_{i+1}x) & i \geq 0 \\ * &= t(s_0 x) & \end{aligned}$$

Suppose (G, t, K) is a principal bundle. The **associated bundle complex** $G \times_t K$ has the same underlying set as $G \times K$ and same structure maps except for d_0 which "twists" via t, i.e.

$$d_0(a, x) = (d_0 a \cdot t(x), d_0(x)).$$

Proposition 4.3.1. *The bundle complex is a simplicial set and* $G \to G \times_t K \xrightarrow{p} K$ *is a fibration.*

See [May67, Proposition 18.4] for a proof.

Definition 4.3.2. If the bundle complex $G \times_t K$ of a principle bundle (G, t, K) is contractible then G is called a **loop group** for K. Then K is called a **classifying complex** for G.

A straightforward application of the long exact sequence applied to the fibration from Proposition 4.3.1 shows the following result.

Lemma 4.3.1. *If* GK *is a loop group for* K *then* $\pi_n K = \pi_{n-1} GK$.

Thus GK does indeed "homotopically look like" the usual loop space construction.

Proposition 4.3.2. *Let* K *be a reduced simplicial set. Then* GK *is a loop group, i.e.* $GK \times_t K$ *is contractible, where* $t : K \to GK$ *is the twisting function* $x \mapsto [x]$.

To prove this proposition one must show that $EK = GK \times_t K$ contractible. This may be done by showing that EK is connected, simply connected, and acyclic (in the sense that all homology groups vanish). Then using a Hurewicz argument and the fact that EK is fibrant, one concludes that EK is contractible. See [Cur67, Theorem 3.16] for a proof.

4.4 A different total space

In the previous section we saw that $EK = GK \times_t K$ is contractible when K is a reduced simplicial set, thereby showing that GK is a loop group *for K*. We now show that GK is also a loop group for $\overline{W}GK$, i.e. there is a principal bundle $(GK, s, \overline{W}GK)$ such that the total space $GK \times_s \overline{W}GK$ is again contractible.

Lemma 4.4.1. *Given any simplicial group G, the map $s : \overline{W}G \to G$, $(g_{n-1}, \cdots, g_0) \mapsto g_{n-1}$ is a twisting function.*

Proof. We must show that the identities from Definition 4.3.1 hold. Showing $d_i s = s d_{i+1}$ and $s_i s = s s_{i+1}$ $(i > 0)$ is straightforward. The other two identities are checked as follows.

$$\begin{aligned} s(d_1(\overline{g}) \cdot (s(d_0(\overline{g}))^{-1} &= s(d_0 g_{n-1} \cdot g_{n-2}, g_{n-3}, \ldots, g_0) \cdot (s(g_{n-2}, \ldots, g_0))^{-1} \\ &= d_0 g_{n-1} \cdot g_{n-2} \cdot g_{n-2}^{-1} \\ &= d_0 g_{n-1} \\ &= d_0 s(g_{n-1}, \ldots, g_0) \end{aligned}$$

And

$$s(s_0(\overline{g})) = s(e_n, \overline{g}) = e_n.$$

□

Define $s : \overline{W}GK \to GK$ by

$$([x_{n-1}], \cdots, [x_0]) \longmapsto [x_{n-1}]$$

From the lemma it follows that s is a twisting function, thus $(GK, s, \overline{W}GK)$ is a principal bundle.

Proposition 4.4.1. *The total space $GK \times_s \overline{W}GK$ is contractible, thus GK is a loop group for $\overline{W}GK$.*

4.5 Unit and counit are weak equivalences

In this and the next section we prove that the unit and counit

$$\begin{aligned} \eta : id_{\mathbf{sSet}_0} &\longrightarrow \overline{W}G \\ \varepsilon : G\overline{W} &\longrightarrow id_{\mathbf{sGrp}} \end{aligned}$$

of the adjunction $G \dashv \overline{W}$, are weak equivalences with respect to the relevant structure.

We now sketch Kan's original argument for the following result ([Kan58, Theorem 11.2]).

Proposition 4.5.1. *If K is a reduced simplicial set the the simplicial map $\eta : K \to \overline{W}GK$ is a weak homotopy equivalence.*

Proof. (Sketch) Consider the following diagram

$$\begin{array}{ccc} K & \xrightarrow{\eta} & \overline{W}GK \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\ GK & \xrightarrow{id} & GK \end{array}$$

where s and t are the twisting functions defined in Section 4.4 and Section 4.3, respectively. One can check that the diagram commutes. If follows that there is a well-defined morphism of simplicial sets $j : GK \times_t K \to GK \times_s \overline{W}GK$ given by $j([x], \sigma) = ([x], \eta\sigma)$. Now the diagram

$$\begin{array}{ccccc} GK & \xrightarrow{q} & GK \times_t K & \xrightarrow{p} & K \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \eta} \\ GK & \xrightarrow{q'} & GK \times_s \overline{W}GK & \xrightarrow{p'} & \overline{W}GK \end{array}$$

where the rows are Kan fibrations (cf. Proposition 4.3.1). One can check that the diagram is commutative, i.e. is a map of Kan fibrations. Thus it induces a map between the long exact sequences associated to these fibrations (cf. [GJ99, chap. 1, Lemma 7.3] for the simplicial version of the long exact sequence associated to a fibration). Since both $GK \times_t K$ and $GK \times_s \overline{W}GK$ are contractible, the map j, trivially induces an isomorphism on homotopy groups, as does id_{GK}. Thus by the five lemma applied to the map of long exact sequences we see that η induces an isomorphism on homotopy groups as well. □

As similar argument gives the corresponding result for the counit.

Proposition 4.5.2. *If A is a simplicial group, then the counit $\varepsilon : G\overline{W}A \to A$ is a weak equivalence in the*

See [Kan58, Proposition 11.3].

4.6 Model structure on simplicial groups

In the rest of this chapter we put a model structure on the category of r-reduced simplicial groups and show that the pair $(G, \overline{W})$ form a Quillen equivalence between the resulting model category and the one defined in the previous chapter.

In [Qui67, Part II, chap. 3] Quillen put a model structure on **sGrp** which we briefly describe. Let us recall Moore's definition of the homotopy groups of a simplicial

group H, these are define by taking the homology of the normalized complex N_*G [May67, chap. 4, sec. 17, in particular Theorem 17.4].
We make the following definitions for a map f in **sGrp**:

(i) The map f is a *weak equivalence* if the induced map $\pi_*(f)$ on the (Moore) homotopy groups, is an isomorphism.
(ii) The map f is a *fibration* if the maps $N_q f$ are surjective for all positive q
(iii) The map f is a *cofibration* if it is a retract of almost free simplicial group maps (cf. Appendix A.4)

Remark 4.6.1. In fact the results of Section 5.3 can be used to give **sGrp** a model structure. However it takes a little work to show that the two structure are in fact the same (one can use [Qui67, Part II, chap. 3, Proposition 1]). As a result, a map in **sGrp** is a fibration (in the above sense) if and only if it is a Kan fibration on the underlying simplicial set level.

Apart from this model structure we shall also rely on the following "left-properness" result due to Quillen [Qui69, Part II, chap 3].

Theorem 4.6.1. *Let*

$$\begin{array}{ccc} H & \xrightarrow{i} & G \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ H' & \xrightarrow[i']{} & G' \end{array}$$

be a pushout square in **sGrp** *where either i of f is a cofibration. If f is a weak equivalence, so is f'.*

The proof of Theorem 4.8.2 uses the theory of group homology with local coefficients. See [Qui69, pp. 249-252] for a proof.

4.7 r-Reduced simplicial groups

Definition 4.7.1. A simplicial group H is called r-**reduced** if the groups $H_0, \cdots, H_r$ are all trivial. A 0-reduced simplicial group is called a **reduced** group. In particular H is reduced if H_0 is the trivial group. For $r \geq 0$ we let $\mathbf{sGrp}_r$ denote the full subcategory of **sGrp** consisting of r-reduced simplicial groups.

As was the case with r-reduced simplicial sets, $\mathbf{sGrp}_r$ is a both reflective and coreflective subcategory of **sGrp**. One can use analogues of the reduction functor $\mathbf{red}_r$ and Eilenberg subcomplex functor $E_r(-)$ to prove this (see Section 5.4). As a result we have the following proposition.

Proposition 4.7.1. *The category* $\mathbf{sGrp}_r$ *is both complete and cocomplete. All limits and colimits may be calculated by composing with the inclusion functor into* **sGrp**.

4.8 Model structure on $\mathbf{sGrp}_r$

Note that if H is a reduced simplicial group, then it is connected, i.e. $\pi_0 H = 0$.

Definition 4.8.1. A map $f : H \to H'$ of connected simplicial groups such that

$$\pi_* f \otimes \mathbb{Q} : \pi_* H \otimes \mathbb{Q} \to \pi_* H' \otimes \mathbb{Q}$$

is an isomorphism, is called a *rational equivalence* or sometimes a $\mathbb{Q}$-*equivalence*.

Definition 4.8.2. Let f be a morphism in $\mathbf{sGrp}_r$ for $r \geq 0$. We make the following definitions.

(i) Call f a *weak equivalence* if it is a $\mathbb{Q}$-equivalence (cf. Definition 4.8.1).
(ii) Call f a *cofibration* if it is a cofibration with respect to the model structure on **sGrp**
(iii) Call f a *fibration* if it has the RLP with respect to all the acyclic cofibrations (i.e. the injective $\mathbb{Q}$-equivalence).

Remark 4.8.1. Note that a map f in $\mathbf{sGrp}_r$ which is a fibration in **sGrp** is also a fibration in $\mathbf{sGrp}_r$.

Theorem 4.8.1. *The category* $\mathbf{sGrp}_r$ *with the proposed model structure of Definition 4.8.2 is a model category, which we denote by* $\mathbf{sGrp}_r^{\mathbb{Q}}$.

Proposition 4.7.1 shows that $\mathbf{sGrp}_r$ is a complete and cocomplete category. The 2-out-of-3 axiom for weak equivalences is clear, as is the retract axiom. One of the lifting axioms is also clear; by the definition of fibrations, any acyclic cofibration has the LLP with respect to fibrations. It remains to prove that acyclic fibrations have the RLP with respect to cofibrations and to prove the factorization axiom.
The proof uses the following analogue of Theorem 4.8.2.

Theorem 4.8.2. *Let*

$$\begin{array}{ccc} H & \xrightarrow{\ i\ } & G \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ H' & \xrightarrow[\ i'\ ]{} & G' \end{array}$$

be a pushout square in **sGrp** *where either i of f is a cofibration and G is connected. If f is a* $\mathbb{Q}$*-equivalence, so is* f'.

As was the case with Theorem 4.8.2 we will not prove this theorem. See [Qui69, p. 252] for the proof.
We can now use the work done in Section 3.1 to help create the required factorizations.

Proposition 4.8.1. *Any map $f : H \to H'$ in $\mathbf{sGrp}_r$ may be factored as $f = pi$ where i is a cofibration and p is an acyclic fibration.*

Proof. Consider the the map $\overline{W}f : \overline{W}H \to \overline{W}H'$ in $\mathbf{sSet}_{r+1}$. By the model structure (Definition 3.2.2) on $\mathbf{sSet}^{\mathrm{Q}}_{r+1}$, factor $\overline{W}f$ as

$$\overline{W}H \overset{u}{\hookrightarrow} X \underset{\sim}{\overset{v}{\twoheadrightarrow}} \overline{W}H'$$

where u is a cofibration in $\mathbf{sSet}^{\mathrm{Q}}_{r+1}$ and v is an acyclic fibration in $\mathbf{sSet}^{\mathrm{Q}}_{r+1}$. Thus u is injective and v is a surjective weak equivalence of simplicial sets. Apply G to this factorization and consider the diagram

$$\begin{array}{ccccc}
G\overline{W}H & \xrightarrow{Gu} & GX & \xrightarrow{Gv} & G\overline{W}H' \\
\downarrow{\scriptstyle \varepsilon_H} & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle \varepsilon_{H'}} \\
H & \xrightarrow[i]{} & Z & \xrightarrow[p]{} & H'
\end{array}$$

where the first square is a pushout, and p is the map induced by f and $\varepsilon_{H'} \circ Gv$. Now Gu is a cofibration in $\mathbf{sGrp}^{\mathrm{Q}}_r$ since it is a cofibration in $\mathbf{sGrp}$ (since it is an almost free map). By Theorem 4.8.2 i' is a Q-equivalence. The counit maps ε_H and $\varepsilon_{H'}$ are weak equivalences in $\mathbf{sGrp}$ (Proposition 4.5.2) hence also in $\mathbf{sGrp}_r$, thus by the 2-out-of-3 property, p is a Q-equivalence too. The counit maps are surjective and so by the commutativity of the second square, p is surjective, hence is a fibration in $\mathbf{sGrp}$ (since surjective maps are fibrations in $\mathbf{sGrp}$, [Qui67, Part II, p. 3.10]) and so also in $\mathbf{sGrp}^{\mathrm{Q}}_r$ (cf. Remark 4.8.1). The map i is a cofibration since it is a pushout of a cofibration (in $\mathbf{sGrp}$). Thus $f = pi$ gives a required factorization. □

Note that the map p we constructed is a fibration in $\mathbf{sGrp}$! As a corollary we get the remaining lifting axiom.

Corollary 4.8.1. *Acyclic fibrations in $\mathbf{sGrp}^{\mathrm{Q}}_r$ have the RLP with respect to cofibrations.*

Proof. This is an application of the retract argument (Lemma 2.3.1). Suppose f is an acyclic fibration. Using Proposition 4.8.1, factor f as $f = pi$ where p is an acyclic fibration in $\mathbf{sGrp}$ and i is a cofibration. By the 2-out-of-3 property, i is an acyclic cofibration, hence f has the RLP with respect to i. Thus f is a retract of p and so has the RLP with respect to cofibrations from $\mathbf{sGrp}$ hence *a fortiori* in $\mathbf{sGrp}_r$. □

We also have the following characterizations of the acyclic fibrations in $\mathbf{sGrp}^{\mathrm{Q}}_r$ ([Qui69, Part II, Proposition 3.3.]).

Corollary 4.8.2. *The acyclic fibrations in $\mathbf{sGrp}^{\mathrm{Q}}_r$ are exactly those maps in $\mathbf{sGrp}_r$ which are surjective weak equivalences.*

It remains to provide acyclic cofibration- fibration factorizations in $\mathbf{sGrp}_r^{\mathbb{Q}}$. We now set out to give these.

Lemma 4.8.1. *Any map $f : H \to H'$ in $\mathbf{sGrp}_r$ may be factored as $f = pi$ where i is an acyclic cofibration and p is a map such that $N_q(p)$ is surjective for $q > r+1$, $\pi_*\mathrm{Ker}(p)$ is uniquely divisible and $\mathrm{Coker}\pi_{r+1} f$ is torsion-free.*

Proof. Suppose first that $\pi_{r+1} f \otimes \mathbb{Q}$ is surjective. Factor $\overline{W} f$ in $\mathbf{sSet}_{r+1}^{\mathbb{Q}}$ as

$$\overline{W}H \underset{\sim}{\overset{u}{\hookrightarrow}} K \overset{v}{\twoheadrightarrow} \overline{W}H'$$

where u is an acyclic cofibration (i.e. an injective $\mathbb{Q}$-equivalence) and v a Kan fibration, in $\mathbf{sSet}_{r+1}^{\mathbb{Q}}$. By Lemma 3.2.7 we get that v is a Kan fibration in $\mathbf{sSet}_*$ and that $\pi_*\mathrm{Ker} v$ is uniquely divisible (here we use that $\pi_{r+1}\overline{W} f \otimes \mathbb{Q}$ is surjective, i.e. $\overline{W}$ preserves this property).
We now apply Kan's loop group functor G to get back to $\mathbf{sGrp}_r$. Since v is a Kan fibration (between connected simplicial sets) it is surjective, thus Gv is surjective too. Now $\overline{W}G$ preserves fibrations and so we have a map of fibrations, i.e. a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Ker} v & \longrightarrow & K & \xrightarrow{\ v\ } & \overline{W}H' \\
\downarrow & & \downarrow{\scriptstyle \eta_K} & & \downarrow{\scriptstyle \eta_{\overline{W}H'}} \\
\overline{W}\mathrm{Ker} Gv & \longrightarrow & \overline{W}GK & \xrightarrow{\ \overline{W}Gv\ } & \overline{W}G\overline{W}H'
\end{array}$$

Considering the induced map on the long exact homotopy sequences we see that $\pi_{q+1}(\overline{W}\mathrm{Ker} Gv) \cong \pi_{q+1}(\mathrm{Ker} G)$. Since $\overline{W}$ is delooping functor (Lemma 4.3.1) we have $\pi_{q+1}(\overline{W}\mathrm{Ker} Gv) \cong \pi_q \mathrm{Ker} Gv$. Thus $\mathrm{Ker} Gv$ has uniquely divisible homotopy groups.
We form the diagram

$$\begin{array}{ccccc}
G\overline{W}H & \longrightarrow & GK & \xrightarrow{\ Gv\ } & G\overline{W}H' \\
\downarrow{\scriptstyle \varepsilon_H} & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle \varepsilon_{H'}} \\
H & \xrightarrow{\ i\ } & Z & \xrightarrow{\ p\ } & H'
\end{array}$$

such that the first square is a pushout and p is induced by f and $\varepsilon_{H'} \circ Gv$. The map $G\overline{W}H \to GK$ is a cofibration (since it is almost free) and the counit ε_H is a weak equivalence (Proposition 4.5.2). By Theorem 4.8.2, i' is a weak equivalence in $\mathbf{sGrp}_r^{\mathbb{Q}}$. Since $\varepsilon_{H'} \circ Gv$ is surjective so is p. By the commutativity of the right-hand square, $\pi_*\mathrm{Ker} p \cong \pi_*\mathrm{Ker} Gv$ which we have just show is uniquely divisible. Thus $f = pi$ gives the required factorization.
The case where $\pi_{r+1} f \otimes \mathbb{Q}$ is not surjective can, through some work, be reduced to the case thus far treated. We will not give the details of this reduction here (see [Qui69, Part II, chap 3, p. 247]). □

Proposition 4.8.2. *The following conditions are equivalence for a map f in $\mathbf{sGrp}_r$.*

(i) *f is a fibration in $\mathbf{sGrp}_r^{\mathbb{Q}}$.*
(ii) *$N_q(f)$ is surjective for $q > r+1$, $\pi_*\mathrm{Ker}(p)$ is uniquely divisible and $\mathrm{Coker}\pi_{r+1}f$ is torsion-free.*

Proof. $(i) \Rightarrow (ii)$: Suppose f is a fibration in $\mathbf{sGrp}_r^{\mathbb{Q}}$. Using Lemma 4.8.1 we factor f as $f = pi$ with i an acyclic cofibration and p a map which satisfies condition (ii). Since f is a fibration it has the RLP with respect to i. Thus f is a retract of p. It is straightforward to see (by writing out the required diagrams) that a retract of a map satisfying condition (ii) also satisfies condition (ii).
$(ii) \Rightarrow (i)$: This direction depends on the details of the general part of the proof of Lemma 4.8.1. □

Proposition 4.8.2 together with Lemma 4.8.1 show that we may factor any map f in $\mathbf{sGrp}_r^{\mathbb{Q}}$ as an acyclic cofibration followed by a fibration. This completes the proof of the model category axioms, and thereby the proof of Theorem 4.8.1.
Using Proposition 2.4.6 we can deduce the following characterization of $\mathrm{Ho}(\mathbf{sGrp}_r^{\mathbb{Q}})$.

Theorem 4.8.3. *The homotopy category $\mathrm{Ho}(\mathbf{sGrp}_r^{\mathbb{Q}})$ is equivalent to the category whose objects are the r-reduced almost free simplicial groups with uniquely divisible homotopy groups and with homotopy classes of maps of simplicial groups.*

4.9 Reduced simplicial groups and reduced simplicial sets

We have succeeded in putting model structures on the categories $\mathbf{sSet}_{r+1}$ and $\mathbf{sGrp}_r$. In this section we show that the functors G and $\overline{W}$ induce an equivalence of homotopy categories $\mathrm{Ho}(\mathbf{sSet}_{r+1}^{\mathbb{Q}})$ and $\mathrm{Ho}(\mathbf{sGrp}_r^{\mathbb{Q}})$.

Theorem 4.9.1. *The adjoint functors*

$$\mathbf{sSet}_{r+1}^{\mathbb{Q}} \underset{\overline{W}}{\overset{G}{\rightleftarrows}} \mathbf{sGrp}_r^{\mathbb{Q}}$$

are Quillen equivalences (for $r \geq 0$). In particular they induce an equivalence of categories $\mathrm{Ho}(\mathbf{sSet}_{r+1}^{\mathbb{Q}}) \cong \mathrm{Ho}(\mathbf{sGrp}_r^{\mathbb{Q}})$.

Proof. Proposition 4.2.1 shows that G and $\overline{W}$ are indeed adjoint. Furthermore, by Proposition 4.5.2 and Proposition 4.5.2 both G and $\overline{W}$ preserve all weak equivalence (they already do so on integral homotopy groups). The second claim follows from Theorem 2.6.2. □

Chapter 5

Model Structures on Simplicial "Algebraic" Categories

The purpose of this chapter is (a) to introduce simplicial model categories and (b) to prove a general result giving sufficient conditions on a category $\mathcal{C}$ ensuring that $s\mathcal{C}$ has a simplicial model structure. This exposition follows [Hir03] and [GJ99] quite closely.

5.1 Simplicial categories

We define *simplicial categories* by which we mean a category which is enriched over **sSet**, as opposed to, say, a simplicial object in the category **CAT** of categories.

Definition 5.1.1. A **simplicial category** $\mathcal{C}$ is a category together with the following structure.

(i) For each pair of objects X and Y of $\mathcal{C}$ there is a simplicial set $\mathbf{Map}(X, Y)$ called the **simplicial mapping space** of X and Y.
(ii) For objects X, Y and Z of $\mathcal{C}$ there is a map of simplicial sets

$$c_{X,Y,Z} : \mathbf{Map}(Y, Z) \times \mathbf{Map}(X, Y) \to \mathbf{Map}(X, Z)$$

called the **composition rule**.
(iii) For every object X there is a simplicial map $i_X : * \to \mathbf{Map}(X, X)$ (where $*$ is the terminal object in **sSet**)
(iv) For objects X and Y of $\mathcal{C}$, an isomorphism $\mathbf{Map}(X, Y)_0 \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$ between morphisms in $\mathcal{C}$ and 0-simplices in $\mathbf{Map}(X, Y)$.

This structure is subject to "associativity" and "unital" requirements, namely that the following diagrams commute.

$$\begin{array}{ccc}
(\mathbf{Map}(Y,Z) \times \mathbf{Map}(X,Y)) \times \mathbf{Map}(W,X) & \xrightarrow{c_{X,Y,Z} \times id} & \mathbf{Map}(X,Z) \times \mathbf{Map}(W,X) \\
\downarrow \cong & & \\
\mathbf{Map}(Y,Z) \times (\mathbf{Map}(X,Y) \times \mathbf{Map}(W,X)) & & \downarrow c_{W,X,Z} \\
\downarrow id \times c_{W,X,Y} & & \\
\mathbf{Map}(Y,Z) \times \mathbf{Map}(W,Y) & \xrightarrow{c_{W,Y,Z}} & \mathbf{Map}(W,Z)
\end{array}$$

expressing the associativity.

$$\begin{array}{ccc}
* \times \mathbf{Map}(X,Y) & \xrightarrow{i_Y \times id} & \mathbf{Map}(Y,Y) \times \mathbf{Map}(X,Y) \\
& {\searrow}^{\cong} \quad {\swarrow}_{c_{X,Y,Y}} & \\
& \mathbf{Map}(X,Y) &
\end{array}$$

expressing left unitality. Finally

$$\begin{array}{ccc}
\mathbf{Map}(X,Y) \times * & \xrightarrow{id \times i_X} & \mathbf{Map}(X,Y) \times \mathbf{Map}(X,X) \\
& {\searrow}^{\cong} \quad {\swarrow}_{c_{X,X,Z}} & \\
& \mathbf{Map}(X,Y) &
\end{array}$$

expresses right unitality.

Notation 3. The mapping space will sometimes be given a subscript $\mathbf{Map}_{\mathcal{C}}(X,Y)$ to help indicate that X and Y are objects of $\mathcal{C}$.

Definition 5.1.2. A **closed simplicial category** is a simplicial category $\mathcal{C}$ in which for each pair of objects X and Y of $\mathcal{C}$ and every simplicial set K there are objects $X \otimes K$ and Y^K in $\mathcal{C}$, and natural isomorphisms of simplicial sets

$$\mathbf{Map}(X \otimes K, Y) \cong \mathbf{Map}_{\mathbf{sSet}}(K, \mathbf{Map}(X,Y)) \cong \mathbf{Map}(X, Y^K).$$

Remark 5.1.1. What we are calling a *closed simplicial category* some authors (e.g. [GJ99]) call a *simplicial category*. Our definition of *simplicial category* is consistent with the general concept of a category enriched over another category, in this case **sSet**. The reasoning behind the extra adjective "closed" in Definition 5.1.2 is to mimic the definition of Cartesian *closed* categories.

Proposition 5.1.1. *Suppose $\mathcal{C}$ is a simplicial category. Then the mapping space* $\mathbf{Map}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{sSet}$ *is a functor. If $\mathcal{C}$ is a closed simplicial category, then "tensor" $-\otimes- : \mathcal{C} \times \mathbf{sSet} \to \mathcal{C}$ and $(-)^{-} : \mathcal{C} \times \mathbf{sSet} \to \mathcal{C}$ are functors.*

See [Hir03] for a proof.

Simplicial Functors

Lemma 5.1.1. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are (closed) simplicial categories and $G : \mathcal{D} \to \mathcal{C}$ is a functor with left adjoint $F : \mathcal{D} \to \mathcal{C}$. Suppose that, for every $K \in$ **sSet** and $A \in \mathcal{C}$ there is a natural isomorphism $F(A \otimes K) \cong F(A) \otimes K$. Then the adjunction internalizes, in the sense that there is a natural isomorphism of simplicial sets*

$$\mathbf{Map}_{\mathcal{D}}(FA, B) \cong \mathbf{Map}_{\mathcal{C}}(A, GB).$$

Furthermore G respects the "exponential" structure in the sense that $G(X^K) \cong (GX)^K$ naturally in K and B.

Proof. The proof consists of manipulating the various adjunctions. See [GJ99, chap. 2 sec. 3] for a proof. □

5.2 Simplicial model categories

Definition 5.2.1. A **(closed) simplicial model category** is a model category $\mathcal{C}$ which is also a closed simplicial category such that the following compatibility axiom is satisfied:

SM7 If $i : A \hookrightarrow B$ is a cofibration in $\mathcal{C}$ and $p : X \twoheadrightarrow Y$ is a fibration in $\mathcal{C}$, then the map of simplicial sets

$$\mathbf{Map}(B, X) \xrightarrow{i^* \times p_*} \mathbf{Map}(A, X) \times_{\mathbf{Map}(A,Y)} \mathbf{Map}(B, Y)$$

is a Kan fibration. The map $i^* \times p_*$ is a weak equivalence if either i or p is a weak equivalence.

Remark 5.2.1. The axiom **SM7** generalizes the *homotopy lifting and extension property* ("HLEP") from ordinary homotopy theory of spaces and simplicial sets.

Proposition 5.2.1. *The simplicial model category axiom **SM7** implies the lifting axiom for model categories. More precisely, given a lifting problem*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

*in a simplicial category $\mathcal{C}$ satisfying **SM7**, then if either i or p is a weak equivalence, there is a lift $B \to X$.*

The proof is straightforward and left to the reader.
A closed simplicial category $\mathcal{C}$ has several different "hom-like" objects living in the three (possibly quite different) categories, Set, **sSet** and $\mathcal{C}$. There is the usual $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ consisting of the set of morphisms from X to Y. There is the mapping space $\mathbf{Map}(X, Y)$ which is a simplicial set. There is also the "exponential" X^K which is again an object of $\mathcal{C}$. The simplicial model category axiom **SM7** above is a statement about the second "hom-like" concept, namely $\mathbf{Map}(-,-)$. The next proposition says that one can reinterpret **SM7** with respect to the other two to get equivalent conditions for $\mathcal{C}$ to be a simplicial model category.

Proposition 5.2.2. *Suppose $\mathcal{C}$ is both a model category and a closed simplicial category. Then the following conditions are equivalent.*

*(i) The axiom **SM7** is satisfied, i.e. $\mathcal{C}$ is a simplicial model category.*
(ii) If $i : A \hookrightarrow B$ is a cofibration in $\mathcal{C}$ and $j : L \hookrightarrow K$ is a cofibration of simplicial sets (i.e. a level-wise injection) then the map

$$A \otimes K \coprod_{A \otimes L} B \otimes L \longrightarrow B \otimes K$$

is a cofibration in $\mathcal{C}$. It is an acyclic cofibration if either i or j is a weak equivalence.
(iii) If $j : L \hookrightarrow K$ is a cofibration of simplicial sets (i.e. a level-wise injection) and $p : X \twoheadrightarrow Y$ is a fibration in $\mathcal{C}$, then the map

$$X^K \longrightarrow X^L \times_{Y^L} Y^K$$

is a fibration in $\mathcal{C}$. It is an acyclic fibration if either j or p is a weak equivalence.

Proof. The proof is straightforward checking the requisite lifting properties (cf. Lemma 2.3.2) and using the universal properties of the pullbacks and pushouts used to define the given maps. □

Example: simplicial sets

The category **sSet**, of simplicial sets, with the usual model category structure supports a simplicial model structure. As a simplicial category it has the following structure. Let X, Y and K be simplicial sets.

(i) We let $\mathbf{Map}(X, Y)$ be the usual internal mapping space for **sSet** defined by $\mathbf{Map}(X, Y)_n := \mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta^n, Y)$. (This is the internal mapping space that **sSet** gets from its presheaf structure).
(ii) We let $X \otimes K$ be $X \times K$ and X^K be $\mathbf{Map}(K, X)$.

Proposition 5.2.3. *With the simplicial structure above,* **sSet** *becomes a simplicial model category.*

See [GJ99, chap. 1] for proof or [Hov99, chap. 3].

Homotopy theory in simplicial model categories

Simplicial model categories allow for a more easily accessible homotopy theory due to the extra structure. In **sSet** the internal mapping space between objects X and Y is defined to be $\mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta^n, Y)$. In general, replacing $\times$ by $\otimes$ and **sSet** by $\mathcal{C}$ reproduces the mapping space, as we now show.

Proposition 5.2.4. *Let $\mathcal{C}$ be a simplicial model category with objects X and Y. For every $n \geq 0$ there is a natural bijection $\mathbf{Map}(X,Y)_n \cong \mathrm{Hom}_{\mathcal{C}}(X \otimes \Delta^n, Y)$.*

Proof. This basically follows from the Yoneda lemma. We have

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes \Delta^n, Y) \cong \mathbf{Map}(X \otimes \Delta^n, Y)_0$$

by one of the requirements of the mapping space. Now by the axioms of a closed simplicial category $\mathbf{Map}(X \otimes \Delta^n, Y)_0$ is naturally isomorphic to $\mathbf{Map}_{\mathbf{sSet}}(\Delta^n, \mathbf{Map}(X,Y))_0$. By definition of the simplicial structure from Proposition 5.2.3,

$$\mathbf{Map}_{\mathbf{sSet}}(\Delta^n, \mathbf{Map}(X,Y))_0 = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n \times \Delta^0, \mathbf{Map}(X,Y)) \cong \mathbf{Map}(X,Y)_n$$

by Yoneda. □

Lemma 5.2.1. *Let $\mathcal{C}$ be a simplicial model category. For each object X of $\mathcal{C}$ there are natural isomorphisms $X \otimes \Delta^0 \cong X$ and $X^{\Delta^0} \cong X$.*

Proof. Since $\mathrm{Hom}_{\mathcal{C}}(X,Y) \cong \mathrm{Hom}_{\mathcal{C}}(X \otimes \Delta^0, Y)$ for all objects Y from $\mathcal{C}$, the Yoneda lemma implies $X \cong X \otimes \Delta^0$. Likewise for X^{Δ^0}. □

Enriching a category

Given a complete and cocomplete category $\mathcal{C}$ we now see one way of equipping $s\mathcal{C}$, the category of simplicial objects in $\mathcal{C}$, with a closed simplicial structure.
We first define the "tensor" functor, and then use this to define the mapping space functor. Given $A \in s\mathcal{C}$ and $K \in \mathbf{sSet}$ define $(A \otimes K) \in s\mathcal{C}$ by

$$(A \otimes K)_n = \coprod_{x \in K_n} A_n.$$

For $\theta : m \to n$ in Δ we let $\theta^* : (A \otimes K)_n \to (A \otimes K)_m$ be the composition

$$\coprod_{x \in K_n} A_n \xrightarrow{\coprod \theta^*} \coprod_{x \in K_n} A_m \longrightarrow \coprod_{x \in K_m} A_m$$

where the second map acts by sending the component corresponding to $x \in K_n$ to the component corresponding to $\theta^*(x) \in K_m$ along the identity map. This construction is clearly functorial in both variables.

Using $\otimes$ as just defined and Proposition 5.2.4 as a guide, we define the mapping space functor by

$$\mathbf{Map}_{s\mathcal{C}}(X,Y)_n := \mathrm{Hom}_{s\mathcal{C}}(X \otimes \Delta^n, Y).$$

for objects X and Y in $s\mathcal{C}$.

Theorem 5.2.1. *Let $\mathcal{C}$ be a complete and cocomplete category. The construction* **Map** : $\mathcal{C}^{op} \times \mathcal{C} \to$ **sSet** *just described makes $s\mathcal{C}$ a closed simplicial category.*

See [GJ99, chap. 2 Thm. 2.5] for a proof. The proof relies on quite general facts (Yoneda lemma, co-Yoneda lemma) using that **sSet** is a presheaf category.

Remark 5.2.2. Applying Theorem 5.2.1 to the category $\mathcal{C} = \mathrm{Set}$ we get a simplicial structure on **sSet**. This is the usual structure, the one defined in Section 5.2. To see this note that a product of simplicial sets $T \times K$ may be given in degree n as a coproduct

$$(T \times K)_n = T_n \times K_n \cong \coprod_{x \in T_n} K_n.$$

This determines the simplicial structure.

5.3 Finding simplicial model categories

This section is devoted to proving rather general results about when the category of simplicial objects in some fixed category $\mathcal{C}$, may be given a simplicial model structure. These results will be used to show that the category of r-reduced simplicial complete Hopf algebras and the category of r-reduced simplicial Lie algebras, may be equipped with simplicial model structures. We follow [GJ99, chap. 2 sec. 3-5].

Suppose $\mathcal{C}$ is some category equipped with adjoint functors $G : \mathcal{C} \to \mathrm{Set}$ and $F : \mathrm{Set} \to \mathcal{C}$, where F is left adjoint. The prolongated functors $s\mathcal{C} \to$ **sSet** and **sSet** $\to s\mathcal{C}$ will also be denoted by G and F, respectively.

Definition 5.3.1. Let f be a morphism in $s\mathcal{C}$.

(i) Call f a *weak equivalence* if Gf is a weak equivalence in **sSet**.
(ii) Call f a *fibration* if Gf is a fibration in **sSet**.
(iii) Call f a *cofibration* if f has the LLP with respect to all acyclic fibrations in $s\mathcal{C}$.

The theorem that we aim to prove is the following.

Theorem 5.3.1. *Suppose $\mathcal{C}$ is a category, $G : \mathcal{C} \to$ Set and $F :$ Set $\to \mathcal{C}$ are functors. Suppose that the following conditions are satisfied.*

(i) $\mathcal{C}$ is complete and cocomplete.
(ii) F is left adjoint to G.

(iii) In the terms of Definition 5.3.1, every cofibration with the LLP with respect to all fibrations is a weak equivalence.
(iv) The functor G preserves filtered colimits.

Then, $s\mathcal{C}$ is a model category with the structure from Definition 5.3.1. The model category $s\mathcal{C}$ may be enriched with the structure of a closed simplicial model category.

Proof. First note that the completeness assumptions on $\mathcal{C}$ ensure that $s\mathcal{C}$ is complete and cocomplete (limits and colimits are computed pointwise). The 2-out-of-3 axiom for $s\mathcal{C}$ follows from the corresponding statement in **sSet**. Likewise if f is a retract of g and g is a weak equivalence (respectively, a fibration), then f is a weak equivalence (respectively, a fibration) by the retract axiom for **sSet**. The retract axiom for cofibrations follows directly from the definition. It remains to provide factorizations and to prove the lifting properties. This is done in Lemma 5.3.2, Lemma 5.3.3 and Lemma 5.3.4 below. This proves that $s\mathcal{C}$ has the advertised model structure.
On the simplicial side, Theorem 5.2.1 gives $s\mathcal{C}$ a closed simplicial structure. We must show that the axiom **SM7** is satisfied. The simplicial structure thus given to $s\mathcal{C}$ makes it clear that $F(A \times K) \cong F(A) \otimes K$ since F is a left adjoint, hence preserves coproducts. By Lemma 5.1.1 it follows that G preserves the "exponential" objects. Using this we can check Item (iii) of Proposition 5.2.2. Suppose $j : L \hookrightarrow K$ is a cofibration of simplicial sets and $p : X \twoheadrightarrow Y$ is a fibration in $\mathcal{C}$. We must show that

$$X^K \longrightarrow X^L \times_{Y^L} Y^K$$

is a fibration in $s\mathcal{C}$, i.e. that it becomes a fibration in **sSet** after applying G. Since G is right adjoint and preserves the "exponential" this is equivalent to showing that

$$(GX)^K \longrightarrow (GX)^L \times_{(GY)^L} (GY)^K$$

is a fibration in **sSet**. This follows from the fact that **sSet** is a simplicial model category, Proposition 5.2.3. If either j or p is a weak equivalence, so is the above map. This conclude the proof. □

We now turn to proving the several lemmas used in the above proof. First we need an important lemma about simplicial sets. Recall that a simplicial set K is said to be *finite* if it has only finitely many non-degenerate simplices.

Lemma 5.3.1. *Every finite simplicial set is small. In more detail this means that given a finite simplicial set K and given any ω-sequence of maps $X^0 \to X^1 \to \cdots$, the natural map $\mathrm{colim}_n \mathrm{Hom}_{\mathbf{sSet}}(K, X^n) \to \mathrm{Hom}_{\mathbf{sSet}}(K, \mathrm{colim}_n X^n)$ is a bijection.*

See [Hov99, chap. 3 Lemma 3.1.2] for a proof. In particular $\Delta^n, \partial\Delta^n$ and Λ^n_k are all finite.

Lemma 5.3.2. *Under the assumptions of Theorem 5.3.1 any morphism $f : A \to X$ in $s\mathcal{C}$ may be factored as $A \overset{j}{\hookrightarrow} Z \underset{\sim}{\overset{q}{\twoheadrightarrow}} X$ where j is a cofibration and q is an acyclic fibration.*

Proof. We shall define Z as a colimit over an ω-sequence $\{Z_n \to Z_{n+1}\}_{n<\omega}$ where at each stage we define maps $Z_n \overset{j_n}{\to} Z_{n+1}$ and $Z_n \overset{q_n}{\to} X$. We let $Z_0 := A$, and $q_0 : Z_0 \to X$ be f. Given (Z_n, q_n) we construct Z_{n+1} as follows. For each $m \in \omega$ consider the set S_m consisting of all pairs (α, β) such that the following diagram commutes

$$\begin{array}{ccc} F\partial\Delta^m & \xrightarrow{\alpha} & Z_n \\ {\scriptstyle F(i)}\downarrow & & \downarrow{\scriptstyle q_n} \\ F\Delta^m & \xrightarrow{\beta} & X \end{array}$$

Note that S_m is indeed a set. Define Z_{n+1} in the following pushout square

$$\begin{array}{ccc} \coprod_{m\in\omega} \coprod_{(\alpha,\beta)\in S_m} F\partial\Delta^m & \xrightarrow{\sum_m \sum_{S_m} \alpha} & Z_n \\ {\scriptstyle \amalg_m \amalg_{S_m} F(i)}\downarrow & & \downarrow{\scriptstyle j_n} \\ \coprod_{m\in\omega} \coprod_{(\alpha,\beta)\in S_m} F\Delta^m & \longrightarrow & Z_{n+1} \end{array}$$

which exists by our cocompleteness assumption. Note that this defines the map $j_n : Z_n \to Z_{n+1}$. The map $q_{n+1} : Z_{n+1} \to X$ is defined by the universal property of the pushout

$$\begin{array}{ccccc} \coprod_{m\in\omega} \coprod_{(\alpha,\beta)\in S_m} F\partial\Delta^m & \xrightarrow{\sum_m \sum_{S_m} \alpha} & Z_n & & \\ {\scriptstyle \amalg_m \amalg_{S_m} F(i)}\downarrow & & \downarrow{\scriptstyle j_n} & \searrow{\scriptstyle q_n} & \\ \coprod_{m\in\omega} \coprod_{(\alpha,\beta)\in S_m} F\Delta^m & \longrightarrow & Z_{n+1} & & \\ & \searrow{\scriptstyle \sum_m \sum_{S_m} \beta} & & \overset{q_{n+1}}{\dashrightarrow} & X \end{array}$$

In this way we get the ω-sequence $A = Z_0 \overset{j_0}{\to} Z_1 \overset{j_1}{\to} Z_2 \to \cdots$. Let $Z := \mathrm{colim}_n Z_n$ be the colimit over this sequence, again using the cocompleteness assumption.

Let us remark now that the maps j_n are cofibrations: the maps $i : \partial\Delta^m \hookrightarrow \Delta^m$ are cofibrations in **sSet**, it follows easily that $F(i)$ is a cofibration since F is left adjoint to G. It is also easy to check that coproducts and pushouts of cofibrations are cofibrations. Thus, j_n is a cofibration for each n. It follows that the map $j : A \to Z$

(which is the transfinite composition of the j_n's) is also a cofibration. This may be proven directly using the universal property of the colimit to reduce the lifting problem to the case $j_n \circ \cdots \circ j_0 : A \hookrightarrow Z_n$ which is a cofibration.

It remains to define $q : Z \to X$ and show that it is an acyclic fibration. The definition comes from the universal property of the colimit. The maps $(q_n)_{n\in\omega}$ form a family that is compatible with the structure maps $(j_n)_{n\in\omega}$ and thus induces a unique map $q : Z \to X$ commuting with the structure maps of the colimit Z.

To show q is an acyclic fibration we must show Gq is an acyclic fibration in **sSet**. For this it suffices to show that it has the RLP with respect to the maps $\partial\Delta^m \to \Delta^m$. By assumption, G commutes with filtered colimits, and so the natural map $\mathrm{colim}_n GZ_n \xrightarrow{\cong} G(\mathrm{colim}_n Z_n)$ is an isomorphism. Thus we may assume we are given a commuting diagram

$$\begin{array}{ccc} \partial\Delta^m & \xrightarrow{\alpha^\sharp} & \mathrm{colim}_n GZ_n \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle Gq} \\ \Delta^m & \xrightarrow[\beta^\sharp]{} & GX \end{array}$$

where we wish to find h making the filled diagram commute. By Lemma 5.3.1 the natural map

$$\mathrm{Hom}(\partial\Delta^m, \mathrm{colim}_n GZ_n) \overset{\cong}{\leftarrow} \mathrm{colim}_n \mathrm{Hom}(\partial\Delta^m, GZ_n).$$

is a bijection, so $\alpha^\sharp$ factors through the natural map $GZ_n \to \mathrm{colim}_n GZ_n$ for some n. Then we have the commutative diagram

$$\begin{array}{ccccccc} \partial\Delta^m & \xrightarrow{\widetilde{\alpha}^\sharp} & GZ_n & \xrightarrow{Gj_n} & GZ_{n+1} & \longrightarrow & \mathrm{colim}_n GZ_n \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle Gq_n} & & \downarrow{\scriptstyle Gq_{n+1}} & & \downarrow{\scriptstyle Gq} \\ \Delta^m & \xrightarrow[\beta^\sharp]{} & GX & \xrightarrow{id} & GX & \xrightarrow{id} & GX \end{array}$$

where the composition of the top row is $\alpha^\sharp$. Consider the adjoint diagram:

$$\begin{array}{ccccccc} F\partial\Delta^m & \xrightarrow{\widetilde{\alpha}} & Z_n & \xrightarrow{j_n} & Z_{n+1} & \longrightarrow & \mathrm{colim}_n Z_n \\ {\scriptstyle Fi}\downarrow & \overset{h^\flat}{\nearrow} & \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle q_{n+1}} & & \downarrow{\scriptstyle q} \\ F\Delta^m & \xrightarrow[\beta]{} & X & \xrightarrow{id} & X & \xrightarrow{id} & X \end{array}$$

The pair $(\widetilde{\alpha}, \beta)$ is in S_m and so by the pushout construction of Z_{n+1} there is a map $h^\flat : F\Delta^m \to Z_{n+1}$ making the filled diagram commute. The adjoint map h

followed by the natural map $GZ_{n+1} \to \mathrm{colim} GZ_n$ provides the required filling of the original diagram. Thus q is an acyclic fibration. By construction $f = qj$ and so we are done. □

Remark 5.3.1. (Small Object Argument) The proof of Lemma 5.3.2 is an application of the *small object argument* as Quillen calls it in [Qui69, p. 3.4]. The term "small" refers to the fact that $\mathrm{Hom}(\partial\Delta^m, -)$ preserves the sequential colimits, i.e. mapping $\partial\Delta^m$ into the colimit of an ω-sequence is the same as mapping it into some finite stage of the sequence. Grothendieck used a version of the small object argument already in 1957 in his Tohoku paper to prove the existence results concerning injective resolutions in Abelian categories.

Lemma 5.3.3. *Under the assumptions of Theorem 5.3.1 any morphisms $f : A \to X$ in $s\mathcal{C}$ may be factored as $A \overset{j}{\underset{\sim}{\hookrightarrow}} Z \overset{q}{\twoheadrightarrow} X$ where j is a cofibration and q is a fibration. Furthermore j has the LLP with respect to all fibrations. It follows that j is an acyclic cofibration.*

Proof. The argument runs along the same lines as Lemma 5.3.2 only this time we use the acyclic cofibrations $\Lambda^m_k \to \Delta^m$ in **sSet**. Since Λ^m_k is a finite simplicial complex Lemma 5.3.1 still applies and so the small object argument carries through, giving j the LLP with respect to all fibrations. □

It remains to check the lifting axioms for the model structure in order to prove Theorem 5.3.1.

Lemma 5.3.4. *Under the assumptions of Theorem 5.3.1 suppose given a (solid) commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{\beta} & Y \end{array}$$

where i is a cofibration and p is a fibration. If either i or p is a weak equivalence then there exists a morphism $h : B \to X$ making the filled diagram commute.

Proof. If p is a weak equivalence then by assumption the diagram may be filled. Suppose therefore i is an acyclic cofibration and p some fibration. By Lemma 5.3.3 we may factor i as qj where q is a fibration and j has the LLP with respect to all fibrations. By the 2-out-of-3 axiom (which has been verified already) q is a weak equivalence. Thus q has the RLP with respect to i (since i is a cofibration). By the retract argument Lemma 2.3.1 i is a retract of j. Since j has the LLP with respect to all acyclic fibrations, so does i. This completes the proof. □

5.4 Reduction

Quillen's Lie models require, as we have seen, model structures on certain *reduced* versions of the categories involved. In this section we sketch how one can modify the constructions of the previous section to put a model structure on $(s\mathcal{C})_r$, the category of r-reduced simplicial objects in $\mathcal{C}$. For this to make sense we now specify that $\mathcal{C}$ be *pointed* i.e. terminal and initial objects are isomorphic.

Definition 5.4.1. Suppose $\mathcal{C}$ is a complete and cocomplete pointed category. Let $*$ denote a terminal-initial object. A simplicial object X in $s\mathcal{C}$ is said to be r-**reduced** if $X_n \cong *$ for all $n \leq r$. The full subcategory of $s\mathcal{C}$ consisting of the r-reduced objects is denoted by $(s\mathcal{C})_r$.

Definition 5.4.2. Let Z be an object in $s\mathcal{C}$. Define the r-th **Eilbenberg subcomplex** $E_r Z$ of Z as follows. For $n \geq 0$ we consider all injective morphisms $\varphi : r \hookrightarrow n$ in Δ and take the limit over the diagram of monomorphisms $\mathrm{Ker}\varphi^* \hookrightarrow Z_n$ in $\mathcal{C}$. Here $\mathrm{Ker}\varphi^*$ is the fiber of φ^*, defined by the pullback

$$\begin{array}{ccc} \mathrm{Ker}\varphi^* & \hookrightarrow & Z_n \\ \downarrow & & \downarrow \varphi^* \\ * & \hookrightarrow & Z_r \end{array}$$

Thus

$$(E_r Z)_n = \varprojlim_{\varphi \in \mathrm{Hom}_\Delta(r,n),\ \text{injective}} \mathrm{Ker}\varphi^*$$

which comes equipped with a family of morphisms $(E_r Z)_n \to \mathrm{Ker}\varphi^*$ one for each injective map $\varphi : r \to n$. The object $(E_r Z)_n$ then comes with a single monomorphism $(E_r Z)_n \overset{j_n}{\hookrightarrow} Z_n$. Suppose $\theta : s \to t$ is a morphism in Δ. Then for each injective $\varphi : r \to s$ we get a diagram

$$\begin{array}{ccc} (E_r Z)_t & \xrightarrow{j_t} & Z_t \\ \vdots & & \downarrow \theta^* \\ \mathrm{Ker}\varphi^* & \hookrightarrow & Z_n \\ \downarrow & & \downarrow \varphi^* \\ * & \hookrightarrow & Z_r \end{array}$$

which commutes. The universal property of the pullback then gives a unique map $(E_r Z)_t \to \mathrm{Ker}\varphi^*$ making the filled diagram commute. These maps (one for each injective $\varphi : r \to s$) assemble to give a map $(E_r Z)_t \overset{\theta^*}{\to} (E_r Z)_s$ making the following diagram commute

$$\begin{array}{ccc} (E_rZ)_t & \xrightarrow{j_t} & Z_t \\ \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle \theta^*} \\ (E_rZ)_s & \xrightarrow{j_s} & Z_s \end{array}$$

Note that in the case $\mathcal{C} = \mathrm{Set}_*$ (pointed sets) this agree's with the usual definition of the Eilenberg subcomplex. In this case the objects $\mathrm{Ker}\varphi^*$ are the fibers of the maps φ^* over the basepoint and the limit $(E_rZ)_n$ is then the intersection of all these fibers, i.e. the collection of all n-simplices whose faces map to the basepoint.

The Eilenberg subcomplex $E_r(-)$ defines a functor $s\mathcal{C} \to (s\mathcal{C})_r$ which is right adjoint to the inclusion functor $(s\mathcal{C})_r \to s\mathcal{C}$. Similarly one can define a reduction functor which is left adjoint to the inclusion functor. The upshot is that the category $(s\mathcal{C})_r$ is a reflective and coreflective subcategory of $s\mathcal{C}$. This proves the following proposition.

Proposition 5.4.1. *The category $(s\mathcal{C})_r$ is complete and cocomplete.*

Model structure on $(s\mathcal{C})_r$.

Assume as before that $\mathcal{C}$ is a pointed complete and cocomplete category equipped with functors $G : \mathcal{C} \to \mathrm{Set}$ and a left adjoint $F : \mathrm{Set} \to \mathcal{C}$. As usual we prolongate these functors to the associated simplicial categories, using the same notation $G : s\mathcal{C} \to \mathbf{sSet}$, $F : \mathbf{sSet} \to s\mathcal{C}$, $G : (s\mathcal{C})_r \to \mathbf{sSet}_r$ and $F : \mathbf{sSet}_r \to (s\mathcal{C})_r$.
In Section 3.2 we provide a model structure on $\mathbf{sSet}_r$ where the weak equivalences are the *rational* equivalences. In fact this construction still works if one defines the weak equivalences to be morphisms inducing isomorphisms on the usual homotopy groups. We state this result without proof, noting only that it follows entirely the same pattern as the proof for $\mathbf{sSet}_r^{\mathbb{Q}}$ (see [Qui69]).

Theorem 5.4.1. *The category $\mathbf{sSet}_r$ of r-reduced simplicial sets $(r \geq 0)$ has the following model structure.*

(i) A map f is a weak equivalence if it induces isomorphisms on all homotopy groups.
(ii) A map f is a cofibration if it is injective.
(iii) A map f is a fibration if it has the RLP with respect to acyclic cofibrations.

As in Section 3.2 we have the following result.

Proposition 5.4.2. *The acyclic fibrations in $\mathbf{sSet}_r$ are precisely those maps in $\mathbf{sSet}_r$ which are acyclic fibrations in $\mathbf{sSet}$.*

Definition 5.4.3. (Model structure on $(s\mathcal{C})_r$)

(i) A map f in $(s\mathcal{C})_r$ is a *weak equivalence* if $G(f)$ is a weak equivalence in $\mathbf{sSet}_r$.

(ii) A map f in $(s\mathcal{C})_r$ is a *fibration* if $G(f)$ is a fibration in $\mathbf{sSet}_r$.
(iii) a map f in $(s\mathcal{C})_r$ is a *cofibration* if it has the LLP with respect to all acyclic fibrations.

Theorem 5.4.2. *Let $\mathcal{C}$ be a category equipped with functors G and F as above. Assume the following conditions are satisfied*

(i) $\mathcal{C}$ is complete, cocomplete and pointed.
(ii) F is left adjoint to G.
(iii) G preserves filtered colimits.
(iv) $G(Z)$ is a Kan complex (i.e. a fibrant object in **sSet***) for every Z in $s\mathcal{C}$.*

Then the proposed model structure of Definition 5.4.3 satisfies the model category axioms, making $(s\mathcal{C})_r$ into a model category.

By Proposition 5.4.1 $(s\mathcal{C})_r$ is complete and cocomplete. The 2-out-of-3 axiom follows from the corresponding statement in $\mathbf{sSet}_r$. Likewise for the retract axiom. Proposition 5.4.2 immediately implies the following.

Lemma 5.4.1. *The acyclic fibrations in $(s\mathcal{C})_r$ are precisely those maps in $(s\mathcal{C})_r$ which are acyclic fibrations in $s\mathcal{C}$.*

Proposition 5.4.3. *Any morphism f in $(s\mathcal{C})_r$ may be factored as a cofibration followed by an acyclic fibration.*

Proof. The proof is a modification of the proof of Lemma 5.3.2 using the small object argument. We modify the construction of the objects Z_n so that we only take pushouts of the form

$$\begin{array}{ccc} \coprod_{m>r}\coprod_{(\alpha,\beta)\in S_m} F\partial\Delta^m & \xrightarrow{\sum_m \sum_{S_m} \alpha} & Z_n \\ \Big\downarrow{\scriptstyle \amalg_m \amalg_{S_m} F(i)} & & \Big\downarrow{\scriptstyle j_n} \\ \coprod_{m>r}\coprod_{(\alpha,\beta)\in S_m} F\Delta^m & \longrightarrow & Z_{n+1} \end{array}$$

i.e. using only those m such that $m > r$. This works because we only need to achieve lifting properties with respect to the inclusions $\partial\Delta^m \to \Delta^m$ when $m > r$. □

A feature of the proof is that the cofibration produced will still be a cofibration in $s\mathcal{C}$ therefore *a fortiori* a cofibration in $(s\mathcal{C})_r$ (this follows from Lemma 5.4.1). Thus, if f is a cofibration in $(s\mathcal{C})_r$ then factoring $f = pi$ where p is an acyclic fibration and i is a cofibration we see that f is a retract of i hence a cofibration in $s\mathcal{C}$. Thus we have proved.

Lemma 5.4.2. *A map in $(s\mathcal{C})_r$ is a cofibration, acyclic fibration or weak equivalence if and only if it is so as a map in $s\mathcal{C}$.*

So of the different classes of special maps in $(s\mathcal{C})_r$, only the (non-acyclic) fibrations can be different from those of $s\mathcal{C}$.
We now prove the second part of the factorization axiom. It will be useful to note that since G is right adjoint and the Eilenberg subcomplex (cf. Definition 5.4.2) is defined in terms of limits, the canonical map $G(E_rZ) \xrightarrow{\cong} E_rG(Z)$ is an isomorphism.

Proposition 5.4.4. *Any map f in $(s\mathcal{C})_r$ may be factored as an acyclic cofibration followed by a fibration.*

Proof. Let $f : X \to Y$ in $(s\mathcal{C})_r$ be given. Consider f as a map in $s\mathcal{C}$ and factor it as $X \xrightarrow{i'} Z \xrightarrow{p'} Y$ where i' is an acyclic cofibration in $s\mathcal{C}$ and p' is a fibration in $s\mathcal{C}$. Since the Eilenberg subcomplex is right adjoint to the inclusion functor this factorization induces a commutative diagram

$$\begin{array}{ccccc} X & \overset{i'}{\hookrightarrow} & Z & \underset{\sim}{\overset{p'}{\twoheadrightarrow}} & Y \\ & \searrow^{i} & \uparrow & \nearrow_{p} & \\ & & E_rZ & & \end{array}$$

We claim that p is a fibration in $(s\mathcal{C})_r$. We must show that $G(p)$ is a fibration in $\mathbf{sSet}_r$. To do this one must show that given an acyclic cofibration $A \underset{\sim}{\hookrightarrow} B$ in $\mathbf{sSet}_r$ and a lifting problem

$$\begin{array}{ccc} A & \longrightarrow & G(E_rZ) \\ \downarrow & \nearrow & \downarrow{\scriptstyle G(p)} \\ B & \longrightarrow & G(Y) \end{array}$$

there exists a lift. This is straightforward, using that E_r is right adjoint to the inclusion and that $G(E_rZ) \cong E_rG(Z)$.
Now since $G(Z)$ is a Kan complex by assumption, the map $E_rG(Z) \to G(Z)$ is a weak equivalence and so i is a weak equivalence by the 2-out-of-3 property. Using Proposition 5.4.3 we factor i as $i = qj$ where j is a cofibration and q is an acyclic fibration. Then j is in fact also a weak equivalence and so $f = (pq)j$ gives a factorization in terms of an acyclic cofibration followed by a fibration. □

As regards the lifting axiom, the first part is the definition of cofibrations in $(s\mathcal{C})_r$. It remains to show the last part of the lifting axiom.
First a small lemma characterizing acyclic cofibrations between fibrant objects. Recall that in a simplicial model category, the object B^{Δ^1} is a path object i.e. the

map $B \overset{s}{\underset{\sim}{\to}} B^{\Delta^1} \twoheadrightarrow B \times B$ factors the diagonal map (Definition 2.4.2). There are two maps $j_0, j_1 : B^{\Delta^1} \to B$ induced by the projections $pr_0, pr_1 : B \times B \to B$.

Lemma 5.4.3. *If $i : A \underset{\sim}{\hookrightarrow} B$ is an acyclic cofibration in a simplicial model category such that A is fibrant, then i is a strong deformation retract, i.e. there exist retract $r : B \to A$ and a homotopy $h : B \to B^{\Delta^1}$ such that $ri = id_A$, $j_1 h = id_B$, $j_0 h = ir$ and $hi = is$.*

Proof. We get r from the following diagram

$$\begin{array}{ccc} A & \xrightarrow{id} & A \\ {\scriptstyle \sim}\big\downarrow{\scriptstyle i} & \overset{r}{\nearrow} & \big\downarrow \\ B & \longrightarrow & * \end{array}$$

Then h is given by the lift in the diagram

$$\begin{array}{ccc} A & \xrightarrow{si} & B^{\Delta^1} \\ {\scriptstyle \sim}\big\downarrow{\scriptstyle i} & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle (j_0, j_1)} \\ B & \longrightarrow & B \times B \end{array}$$

The stated relations are then easily verified. □

With this general lemma in hand we return to the specific category $(s\mathcal{C})_r$.

Proposition 5.4.5. *Acyclic cofibrations in $(s\mathcal{C})_r$ have the LLP with respect to fibrations.*

Proof. Let $i : A \hookrightarrow B$ be an acyclic cofibration. By Lemma 5.4.2, i is an acyclic cofibration in $s\mathcal{C}$. Since $G(A)$ is assumed to be a Kan complex for all A, Lemma 5.4.3 applies and so r and h exists with the relations noted in the lemma. Suppose $p : X \twoheadrightarrow Y$ is a fibration in $(s\mathcal{C})_r$ and

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle \sim}\big\downarrow{\scriptstyle i} & \nearrow & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{\beta} & Y \end{array}$$

is a lifting problem. Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha^{\Delta^1} s} & E_r(X^{\Delta^1} \\ {\scriptstyle \sim}\big\downarrow{\scriptstyle i} & \overset{k}{\nearrow} & \big\downarrow{\scriptstyle (j_0, E_r(p^{\Delta^1}))} \\ B & \xrightarrow[(\alpha r, \beta^{\Delta^1} h)]{} & X \times_Y E_r(Y^{\Delta^1} \end{array}$$

If the right hand vertical arrow is an acyclic fibration, then the diagram has a lift $k : B \to E_r(X^{\Delta^1})$ and the $\gamma = j_1 k$ is a solution to the original lifting problem. Thus it remains to show that $(j_0, E_r(p^{\Delta^1})$ is an acyclic fibration. See [Qui69, p. 255] for the final part of this argument. □

This completes the proof of Theorem 5.4.2.

Chapter 6

Simplicial Complete Hopf Algebras

In this chapter we introduce the category of complete Hopf algebras, denoted $\mathbf{Hopf}^{\mathrm{comp}}$. There are few sources treating *complete* Hopf algebras in detail. We therefore rely mainly on Quillen's original appendix [Qui69, Appendix A].
In the following section we shall abuse notation, suppressing much of the data involved in our objects. For example a complete augmented algebra $(A, \mu, \eta, \varepsilon, \{F_n A\}_{n\in\omega})$ carries with it several data such as the choice of filtration $\{F_n A\}_{n\in\omega}$. However, we will always drop this choice from the notation, denoting such a gadget simply by A.

6.1 Algebras

All algebras A in this section will be over $k = \mathbb{Q}$. Most of the following would work fine with any other characteristic 0 field. Vector spaces and tensor products etc. will be over $\mathbb{Q}$ unless otherwise is indicated.
An **associative algebra** (A, μ) is a k-vector space A equipped with a linear map $\mu : A \otimes A \to A$ which is associative. We usually write $\mu(a, b) = a \cdot b$ and abuse notation by referring to A as the algebra (A, μ). A **unital** algebra (A, μ, η) is an algebra equipped with a map of algebras $\eta : k \to A$ such that $\eta(1_k) \cdot a = a = a \cdot \eta(1_k)$. If nothing else is noted then "algebra" will always mean associative and unital algebra.

6.2 Complete augmented algebras

A (unital associative) algebra A is **augmented** if has a *counit*, i.e. a map of algebras $\varepsilon : A \to k$ satisfying the dual axioms of those for η. In this case the kernel of ε is denoted $\overline{A}$ and is called the **augmentation ideal** of A. It will play an important

role in what follows. Note that an augmented algebra A splits canonically as $A \cong \overline{A} \oplus k$ (as a k-vector space).
A **filtration** of an algebra A is a decreasing sequence of subspaces

$$A = F_0 A \supseteq F_1 A \supseteq \cdots$$

such that $F_p A \cdot F_q A \subseteq F_{p+q} A$. This last requirement makes $F_p A$ into a two-sided ideal in A for each p.

Example 6.2.1. If A is an augmented algebra with augmentation ideal $\overline{A}$ then $\{\overline{A}^n\}_{n\in\omega}$ is a filtration on A.

Given an algebra A with a filtration, the product μ in A induces a product on the **associated graded algebra**

$$\mathrm{gr} A = \bigoplus_{n=0}^{\infty} F_n A / F_{n+1} A$$

given by multiplying representatives. The associated graded algebra $\mathrm{gr}A$ is very useful when studying properties of A. The n'th filtration quotient with respect to the filtration $\{F_n A\}_{n\in\omega}$ is denoted

$$\mathrm{gr}_n A = F_n A / F_{n+1} A.$$

For the definition of completion with respect to a filtration see Appendix A.3.

Definition 6.2.1. An algebra A together with a filtration $\{F_n A\}_{n\in\omega}$ is called a **complete augmented algebra** provided

(i) The subspace $F_1 A$ is the augmentation ideal $\overline{A}$ of A.
(ii) The associated graded algebra $\mathrm{gr}A$ is generated, as an algebra, by $\mathrm{gr}_1 A$.
(iii) The algebra is *complete* with respect to the filtration, i.e. $A = \varprojlim_n A/F_n A$.

A morphism between two complete augmented algebras is an augmented algebra map $f : A \to A'$ such that $f(F_n A) \subseteq F_n A'$.

Note that condition *(i)* is equivalent with the condition that $\mathrm{gr}_0 A = k$. Furthermore *(i)* implies that $\overline{A}^n \subseteq F_n A$ for all n.

Lemma 6.2.1. *Any augmented algebra B admits a completion making it a complete augmented algebra. This construction is left adjoint to the forgetful functor from complete augmented algebras to augmented algebras.*

See [Qui69, Appendix A, Example 1.2] for a proof of this lemma.

Categorical properties of complete augmented algebras

Example 6.2.2. Let $\{X_i\}_{i\in I}$ be a set of indeterminate (non-commuting) variables indexed over some set $i \in I$ and let $P = k\langle\langle X_i\rangle\rangle_{i\in I}$ denote the algebra of formal power series in the variables X_i. The algebra P is the completion of the algebra $k\langle X_i\rangle$ of polynomials in the X_i with respect to the augmentation map ε which maps X_i to 1 for all i. Therefore P is a complete augmented algebra.

In [Qui69, Appendix A] Quillen shows that the projective objects in the category of complete augmented algebras are exactly those which are isomorphic to $P = k\langle\langle X_i\rangle\rangle_{i\in I}$ for some set I.

Proposition 6.2.1. *Any complete augmented algebra A is the quotient of a power series ring P by a closed ideal.*

Thus, the category of complete augmented algebras has enough projectives. From this structure it is not hard to prove the following completeness and cocompleteness result about complete augmented algebras.

Theorem 6.2.1. *The category of complete augmented algebras is complete and cocomplete and has $P = k\langle\langle x\rangle\rangle$ as a projective generator.*

See [Qui69, Appendix A, Proposition 1.10].

Exponential and logarithmic series

Let A be a complete augmented algebra. The set $1 + \overline{A}$ is a group under multiplication denoted $\mathbf{G}_m A$. Given $a \in \overline{A}$ the inverse of $1 - a$ is the geometric series $1 + a + a^2 + \cdots$, which converges since A is complete and $\overline{A}^n \subseteq F_n A$ for all n. Then, $1 + a = 1 - (-a)$ is also invertible. A map of complete augmented algebras preserves these units so $\mathbf{G}_m$ defines a functor

$$\mathbf{G}_m : \mathbf{Alg}_{\mathrm{aug}}^{\mathrm{comp}} \longrightarrow \mathbf{Grp}.$$

Similarly the augmentation ideal $\overline{A}$ is a Lie algebra with the commutator operation induced by the product on A. We denote this Lie algebra by $\mathbf{G}_a A$. A map of complete augmented algebras preserves the commutator and so induces a map of Lie algebras. Thus $\mathbf{G}_a$ defines a functor

$$\mathbf{G}_a : \mathbf{Alg}_{\mathrm{aug}}^{\mathrm{comp}} \longrightarrow \mathbf{Lie}.$$

Proposition 6.2.2. *The functor $\mathbf{G}_m$ is right adjoint to the completed group algebra functor. The functor $\mathbf{G}_a$ is right adjoint to the completed universal enveloping algebra functor. Thus we have a diagram of adjoints.*

$$\mathbf{Grp} \underset{\mathbf{G}_m}{\overset{\widehat{k}}{\rightleftarrows}} \mathbf{Alg}_{aug}^{comp} \underset{\mathbf{G}_a}{\overset{\widehat{U}}{\leftrightarrows}} \mathbf{Lie}$$

See [Qui69, Appendix A, (1.12)]
Given a complete augmented algebra A with filtration $\{F_n A\}$ we get an induced *group filtration* on $\mathbf{G}_m A$ (see [Ser92, chap. 2, sec. 2] for more on filtrations of groups). The filtration on the group $\mathbf{G}_m A$ is given by

$$F_n \mathbf{G}_m A = 1 + F_n A$$

for $n \geq 1$. The subgroups $F_n \mathbf{G}_m A$ are normal in $F_{n+1} \mathbf{G}_m A$ and satisfy the filtration property, namely that $[F_p \mathbf{G}_m A, F_q \mathbf{G}_m A] \subseteq F_{p+q} \mathbf{G}_m A$ (here the commutator denotes the group commutator, i.e. $[g, h] = ghg^{-1}h^{-1}$). This property ensures that the commutator subgroup $[F_p \mathbf{G}_m A, F_p \mathbf{G}_m A]$ is contained in $F_{p+1} \mathbf{G}_m A$ for all p, thus the filtration quotients

$$F_p \mathbf{G}_m A / F_{p+1} \mathbf{G}_m A$$

are all Abelian. This means that we can define the *associated graded Abelian group*

$$\mathrm{gr}\mathbf{G}_m A = \bigoplus_{n=1}^{\infty} \frac{F_n \mathbf{G}_m A}{F_{n+1} \mathbf{G}_m A}.$$

The commutator operation on $\mathbf{G}_m A$ induces a bilinear operator on $\mathrm{gr}\mathbf{G}_m A$ which makes $\mathrm{gr}\mathbf{G}_m A$ into a Lie algebra over $\mathbb{Z}$ (see [Ser92, chap. 2, Proposition 2.3] for a full proof).
Similarly, there is an induced *Lie algebra filtration* on $\mathbf{G}_a A$ given by

$$F_n \mathbf{G}_a A = F_n A$$

which satisfies the requirement $[F_p \mathbf{G}_a A, F_q \mathbf{G}_a A] \subseteq F_{p+q} \mathbf{G}_a A$. Again we can define the *associated graded Lie algebra*

$$\mathrm{gr}\mathbf{G}_a A = \bigoplus_{n=1}^{\infty} \frac{F_n \mathbf{G}_a A}{F_{n+1} \mathbf{G}_a A}.$$

The Lie bracket on $\mathbf{G}_a A$ induces a Lie bracket on $\mathrm{gr}\mathbf{G}_a A$ making $\mathrm{gr}\mathbf{G}_a A$ a Lie algebra over $\mathbb{Q}$. The function $f : F_n \mathbf{G}_a A \longrightarrow F_n \mathbf{G}_m A$ given by $f(x) = 1 + x$ induces a group homomorphism when taking quotients. This in fact induces an isomorphism of $\mathbb{Z}$-Lie algebras

$$\mathrm{gr}\mathbf{G}_a A \xrightarrow{f} \mathrm{gr}\mathbf{G}_m A.$$

In fact so does any map f given by a power series of the form

$$f(x) = 1 + x + a_2 x^2 + a_3 x^3 + \cdots$$

for $a_n \in \mathbb{Q}$. In particular we can use the exponential power series

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Note that this power series is well defined over k since k has characteristic zero.

Proposition 6.2.3. *The exponential series defines a (set theoretic) function*

$$\exp : \mathbf{G}_a A \longrightarrow \mathbf{G}_m A$$

which is a bijection. The inverse is the logarithm

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n.$$

Proof. (sketch) The exponential series does indeed define a function since for $x \in \overline{A}$ then $x^n \in \overline{A}^n \subseteq F_n A$, thus the power series converges. □

While not a group homomorphism, the exponential does satisfy the relation

$$\exp(x+y) = \exp(x)\exp(y)$$

whenever $[x,y] = 0$ i.e. x and y commute. This follows from the binomial theorem.

6.3 Completed tensor product

In Appendix A.3 we define the completed tensor product on filtered vector spaces. If A and A' are complete augmented algebras with filtrations $\{F_n A\}_n$ and $\{F_n n A'\}_n$ respectively then the (usual) tensor product $A \otimes_k A'$ admits the filtration

$$F_n(A \otimes A') = \bigoplus_{i+j=n} F_i A \otimes F_j A'.$$

The **completed tensor product** of A and A' is the completion $\widehat{A \otimes A'}$ of $A \otimes A'$ with respect to this filtration. We state some result about the completed tensor product of complete augmented algebras. See [Qui69, Appendix A, p. 269].

Lemma 6.3.1. *The completed tensor product of complete augmented algebras is a complete augmented algebra.*

Lemma 6.3.2. *If A and B are augmented algebras then $\widehat{A \otimes B} \cong \widehat{A} \widehat{\otimes} \widehat{B}$.*

Thus if A and B are complete augmented algebras then $A \widehat{\otimes} B$ is a complete augmented algebra.

Proposition 6.3.1. *Suppose A, A' and B are complete augmented algebras and suppose given morphisms $u : A \to B$ and $v : A' \to B$ such that*

$$[ux, vy] = 0 \qquad \text{for all } x \in A \text{ and } y \in A'$$

then there is a unique map

$$w : A \widehat{\otimes} A' \longrightarrow B$$

such that $w(x \widehat{\otimes} 1) = u(x)$ and $w(1 \widehat{\otimes} y) = v(y)$.

6.4 Coalgebras

The study of Hopf algebras encompasses more than the "usual" algebraic structures such as groups, modules and (well...) algebras. A Hopf algebra, apart from being an augmented algebra, is also a *co*algebra. Coalgebras are defined in a way that is dual to the usual definition of algebras, in the sense that "all arrows go in the opposite direction".

Definition 6.4.1. A k-vector space C equipped with k-linear maps $\Delta : C \to C \otimes C$ (called the **coproduct**) and $\varepsilon : C \to k$ (called the **counit**) is called a **coalgebra** if it is *coassociative*, *counital* and *cocommutative*. The meaning of these three requirements is that the following diagrams must commute. Coassociativity is expressed by the following diagram.

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1_C \otimes \Delta} \\ C \otimes C & \xrightarrow[\Delta \otimes 1_C]{} & C \otimes C \otimes C \end{array}$$

Counitality is expressed by the following diagram.

$$\begin{array}{ccccc} & \swarrow^{1_C} & C & \searrow^{1_C} & \\ & & \downarrow{\scriptstyle \Delta} & & \\ C \cong C \otimes k & \xleftarrow[1_C \otimes \varepsilon]{} & C \otimes C & \xrightarrow[\varepsilon \otimes 1_C]{} & k \otimes C \cong C \end{array}$$

Cocommutativity is expressed by the following diagram.

$$\begin{array}{ccc} & C & \\ \swarrow^{\Delta} & & \searrow^{\Delta} \\ C \otimes C & \xrightarrow{\tau} & C \otimes C \end{array}$$

where $\tau : A \otimes B \to B \otimes A$ is the "twist map" determined by $\tau(a \otimes b) = b \otimes a$ (note that we are not in the graded case).

Remark 6.4.1. Unlike for algebras we will always assume that coalgebras are *cocommutative*. Many places in the literature do not assume cocommutativity. The reason we do is that all coalgebras we will deal with have this property.

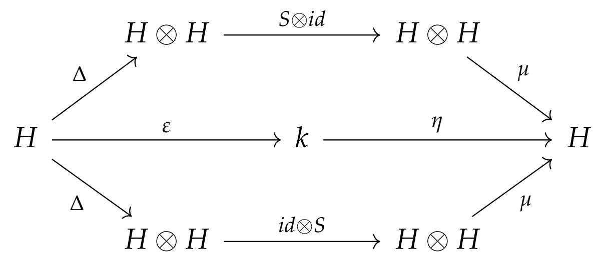
6.5 Hopf algebras

Informally, a Hopf algebra is a set equipped with an (associative, unital) algebra structure, a (coassociative, cocommutative, counital) coalgebra structure, and an antipode map (defined below) such all structures are compatible. To make this more precise we first deal with the combination of algebra and coalgebra structure.

Definition 6.5.1. A **bialgebra** $(H, \mu, \eta, \Delta, \varepsilon)$ over k is a k-vector space H such that (H, μ, η) is a unital algebra, (H, Δ, ε) is a counital cocommutative coalgebra such that Δ is an algebra map (equivalently μ is a coalgebra map) and such that ε is an augmentation (equivalently η is a coagumentation).

We will often denote a bialgebra simply by H or by (H, μ, Δ). The sense in which Δ is required to be an algebra map is the following. We provide $H \otimes H$ with the algebra structure $(H \otimes H) \otimes (H \otimes H) \to H \otimes H$ determined by $\mu_{H\otimes H}((x_1 \otimes y_1) \otimes (x_2 \otimes y_2)) = \mu_H(x_1 \otimes x_2) \otimes \mu_H(y_1 \otimes y_2)$, i.e. multiplication component wise.

Definition 6.5.2. A **Hopf algebra** $(H, \mu, \eta, \Delta, \varepsilon, S)$ over k is a bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ together with a k-linear map $S : H \to H$ called the **antipode** making the following diagram commute.



A bialgebra H has at most one antipode map making it a Hopf algebra. This follows from an equivalent definition of the antipode as the (two-sided) inverse of the identity map $id \in \mathrm{Hom}_k(H, H)$ under the associative *convolution product*. This is a product $\star$ on $\mathrm{Hom}_k(H, H)$ given by $f \star g = \mu \circ (f \otimes g) \circ \Delta$ (see [LV12, chap. 1, sec. 3.10]). Thus, being a Hopf algebra is really a *property* of a bialgebra.

Example 6.5.1. (Group algebra) Let G be a group and kG the group algebra. Then kG is a coalgebra with coproduct determined by $\Delta(g) = g \otimes g$ for $g \in G$. The counit is the augmentation map $\varepsilon : kG \to k$ determined by $\varepsilon(g) = 1$ for all $g \in G$. The antipode $S : kG \to kG$ is determined by $S(g) = g^{-1}$ for $g \in G$. In this case the antipode axiom reduces to the fact that $g^{-1}g = 1 = gg^{-1}$. Note that kG is indeed cocommutative since $\tau(g \otimes g) = g \otimes g$. The Hopf algebra kG is commutative if and only if G is a commutative group.

The Hopf algebra structure is a good example of the importance of noticing algebraic structure. The "isomorphism problem for group algebras" is the following question: if $kG \cong kG'$ as k-algebras does it follow that $G \cong G'$? In general the answer is no; for example if k is the field of complex numbers and $G \not\cong G'$ are two non-isomorphic Abelian groups of equal order then one can show that $kG \cong \mathbb{C}^n \cong kG'$ as k-algebras. The analogous "isomorphism problem for group (Hopf) algebras" is easy to solve; In any Hopf algebra H we say that an element x is **group-like** (see below) if $\Delta(x) = x \otimes x$. Now one can show that for $H = kG$ then the group of group-like elements is exactly G. Furthermore the group of group-like elements forms an invariant of H thus if $kG \cong kG'$ as Hopf algebras then $G \cong G'$ as groups.

Example 6.5.2. (Universal Enveloping Algebra) Let $\mathfrak{g}$ be a Lie algebra over k. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. The map $\Delta(x) = 1 \otimes x + x \otimes 1$ for $x \in \mathfrak{g}$ determines a coproduct on $U(\mathfrak{g})$ with counit $x \mapsto 0$. The antipode $S : U\mathfrak{g} \to U\mathfrak{g}$ is determined by $x \mapsto -x$ for $x \in \mathfrak{g}$.

6.6 Complete Hopf algebras

If A is a complete augmented algebra (cf. Definition 6.2.1) then it can be shown that $A\widehat{\otimes}A$ is also a complete augmented algebra (cf. Lemma 6.3.2), with the filtration described in Section 6.3. So it makes sense to require a map $\Delta : A \to A\widehat{\otimes}A$ to be a morphism of complete augmented algebras.

Definition 6.6.1. A **complete Hopf algebra** $(H, \mu, \eta, \Delta, \varepsilon)$ is a complete augmented algebra H with a coproduct $\Delta : H \to H\widehat{\otimes}H$, which is a map of complete augmented algebras and which is coassociative, cocommutative and such that the augmentation $\varepsilon : H \to k$ is a counit. A k-linear map $f : H \to H'$ of complete Hopf algebras is a *morphism of complete Hopf algebras* if f is a morphism of complete augmented algebras which also preserves the coproduct i.e. $\Delta_{H'} \circ f = (f \otimes f) \circ \Delta$. The category of complete Hopf algebras will be denoted $\mathbf{Hopf}^{\mathrm{comp}}$.

Remark 6.6.1. The definition of a complete Hopf algebras, as given above, does not mention an antipode map. Thus, in modern terminology we might have called these complete *bialgebras* instead. We have, however, decided to stick with Quillen on this matter. Indeed it is the belief of the author that complete Hopf algebras actually do have natural antipode, but we will not go further in this matter.

Lemma 6.6.1. *If H is a cocommutative Hopf algebra then the completion $\widehat{H}$ with respect to the augmentation ideal* $\ker(\varepsilon)$ *admits a complete Hopf algebra structure. The coproduct $\widehat{\Delta} : \widehat{H} \to \widehat{H \otimes H} \cong \widehat{H}\widehat{\otimes}\widehat{H}$ is induced by the coproduct on H and the isomorphism $\widehat{H \otimes H} \cong \widehat{H}\widehat{\otimes}\widehat{H}$, the counit $\widehat{\varepsilon} : \widehat{H} \to \widehat{k} \cong k$ is induced by the augmentation $\varepsilon : H \to k$. This construction is functorial providing a functor $\mathbf{Alg}_{aug}^{comp} \to \mathbf{Hopf}^{comp}$.*

Proof. Let H be any cocommutatuve Hopf algebra. Filter H with powers of the augmentation ideal

$$H = \overline{H}^0 \supseteq \overline{H}^1 \supseteq \overline{H}^2 \supseteq \overline{H}^3 \supseteq \cdots .$$

Recall that we filter the ground field k by $F_0 k = k$ and $F_n k = 0$ for $n > 0$. Then the counit $\varepsilon : H \to k$ preserves the filtration. Likewise for the unit $\eta : k \to H$ (clearly η always preserves the filtration, regardless of the filtration on H). Suppose $x \in \overline{H} = \mathrm{Ker}(\varepsilon)$ is an element of the augmentation ideal. Since ε is a counit the identity $(\varepsilon \otimes id) \circ \Delta = id$ is satisfied and since $\varepsilon(u) = 0$ this implies that $\Delta(u) = 1 \otimes u +$

$y_1 + y_2$ for some $y_1 \in \overline{H} \otimes 1$ and some $y_2 \in \overline{H} \otimes \overline{H}$. The identity $(id \otimes \varepsilon) \circ \Delta = id$ then implies that $\Delta(u) = u \otimes 1 + 1 \otimes u + y$ for some $y \in \overline{H} \otimes \overline{H}$. Thus Δ preserves the first component of the filtration. Using a similar argument (this time using that Δ is an algebra morphism) one shows that Δ preserves the filtration. The product $\mu : H \otimes H \to H$ clearly also preserves the filtration. It now follows that the completion $\widehat{H} = \varprojlim H/\overline{H}^n$ admits the structure of a complete Hopf algebra since the maps $\Delta, \mu, \eta, \varepsilon$ all induce maps satisfying the corresponding identities. Note that to define the diagonal we use the isomorphism $\widehat{H \otimes H} \xrightarrow{\cong} \widehat{H} \widehat{\otimes} \widehat{H}$. □

As a corollary of this canonical completion process, the usual group algebra and universal algebra have completed counterparts.

Corollary 6.6.1. *The completed group algebra $\widehat{k}G$ defines a functor $\widehat{k} : \mathbf{Grp} \to \mathbf{Hopf}^{comp}$. The completed universal enveloping algebra $\widehat{U}\mathfrak{g}$ defines a functor $\mathbf{Lie} \to \mathbf{Hopf}^{comp}$.*

Definition 6.6.2. Let H be a complete Hopf algebra. A subspace $J \subseteq H$ is called a **closed Hopf ideal** provided $\Delta J \subseteq H \widehat{\otimes} J + J \widehat{\otimes} H$.

If $J \subseteq H$ is a closed Hopf ideal then we can take the cokernel H/J. The filtrations on H and J then induce a filtration on H/J and one can then shows that H/J is complete with respect to this filtration.

6.7 Group-like and primitive elements

The notions of group-like (respectively, primitive) elements in a Hopf algebra have proved fruitful. We now make the direct extension of these notions to the case of complete Hopf algebras.

Definition 6.7.1. Let H be a complete Hopf algebra. An element $x \in 1 + \overline{H}$ is said to be **group-like** if $\Delta x = x \widehat{\otimes} x$. The subset of group-like elements is denoted $\mathcal{G}H$. A element $x \in \overline{H}$ is said to be **primitive** if $\Delta x = 1 \widehat{\otimes} x + x \widehat{\otimes} 1$. The set of primitive elements is denoted $\mathcal{P}H$.

So the group like element of H are those $x \in H$ such that $\varepsilon(x) = 1$ and $\Delta(x) = x \widehat{\otimes} x$. The primitive elements are those $x \in H$ such that $\varepsilon(x) = 0$ and $\Delta(x) = x \widehat{\otimes} 1 + 1 \widehat{\otimes} x$.

Lemma 6.7.1. *The set $\mathcal{G}H$ is a subgroup of $\mathbf{G}_m H$. The set $\mathcal{P}H$ is a Lie subalgebra of $\mathbf{G}_a H$. The functor $\mathcal{G} : \mathbf{Hopf}^{comp} \to \mathbf{Grp}$ is right adjont to $\widehat{k}$. The functor $\mathcal{P} : \mathbf{Hopf}^{comp} \to \mathbf{Lie}$ is right adjoint to $\widehat{U}$. Thus we have a diagram of adjoints.*

$$\mathbf{Grp} \underset{\mathcal{G}}{\overset{\widehat{k}}{\rightleftarrows}} \mathbf{Hopf}^{comp} \underset{\mathcal{P}}{\overset{\widehat{U}}{\leftrightarrows}} \mathbf{Lie}$$

Recall the standing assumption that $k = \mathbb{Q}$, in particular k has characteristic zero.

Proposition 6.7.1. *Let H be a complete Hopf algebra. The exponential function* $\exp : H \to H$ *restricts to a bijection $\mathcal{P}H \xrightarrow{\cong} \mathcal{G}H$. This bijection is natural, so we have a natural isomorphism of functors $\mathcal{P} \cong \mathcal{G} : \mathbf{Hopf}^{comp} \to \mathrm{Set}$.*

Proof. Suppose $x \in \mathcal{P}H$. The function $\exp : H \to H$ clearly preserves the coproduct, thus

$$\Delta(\exp(x)) = \exp(\Delta x) = \exp(x\widehat{\otimes}1 + 1\widehat{\otimes}x) = \exp(x\widehat{\otimes}1)\exp(1\widehat{\otimes}x)$$

where the last equality follows from the fact that $1\widehat{\otimes}x$ and $x\widehat{\otimes}1$ commute. Now

$$\exp(x\widehat{\otimes}1)\exp(1\widehat{\otimes}x) = (\exp(x)\widehat{\otimes}1)(1\widehat{\otimes}\exp(x)) = \exp(x)\widehat{\otimes}\exp(x).$$

Also, since $x \in \mathcal{P}H$ we have $\varepsilon(x) = 0$, so $\varepsilon(\exp(x)) = 1$. Thus, $\exp(x)$ is group-like.
For the inverse we use $\log : H \to H$. Once again it is easy to show that $\Delta\log(x) = \log(\Delta x)$. Supposing x to be group-like, we then have $\Delta\log(x) = \log(x\widehat{\otimes}x)$. Using once again that $1\widehat{\otimes}x$ commutes with $x\widehat{\otimes}1$ we have

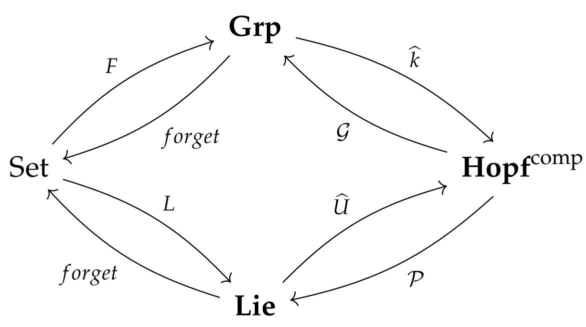
$$\Delta(\log(x)) = \log(x\widehat{\otimes}x) = \log((1\widehat{\otimes}x)\cdot(x\widehat{\otimes}1)) = \log(1\widehat{\otimes}x) + \log(x\widehat{\otimes}1).$$

Now $\log(1\widehat{\otimes}x) = 1\widehat{\otimes}x$ and so the above shows that $\Delta(\log(x)) = 1\widehat{\otimes}\log(x) + \log(x)\widehat{\otimes}1$.
Now since log and exp are inverse power series, we conclude that $\mathcal{P}H \xrightarrow{\cong} \mathcal{G}H$. This isomorphism is clearly natural in H. □

For further investigations of these ideas, and the closely related Baker-Campbell-Hausdorff formula, see [Ser92, chap. 4].

6.8 Free Hopf algebras

We have seen two specific ways to generate complete Hopf algebras: from a group we get the completed group algebra, from a Lie algebra we get the completed universal enveloping algebra. Of course the two Hopf algebras may be very different. However, if we start with just a set then we can consider the free group or the free Lie algebra generated by the set, and then generate the respective complete Hopf algebras. We have the following diagram of functors.



A corollary of Proposition 6.7.1 and Lemma 6.7.1 is that the two ways of generating complete Hopf algebras, from a given set, are isomorphic. We can also give a concrete description of this complete Hopf algebra. Let S be a set. The group algebra $kF(S)$ on the free group generated by S is the algebra of polynomials in non-commuting variables $\{X_s\}_{s\in S}$, denoted $kF(S) = k\langle X_s\rangle_{s\in S}$. This algebra is augmented by the map $\varepsilon(s) = 1$ for $s \in S$ and the completion of this algebra with respect to the augmentation ideal is the ring of powers series in the non-commuting variables $\{X_s\}_{s\in S}$, denoted $k\langle\langle X_s\rangle\rangle_{s\in S}$. We can give $k\langle\langle X_s\rangle\rangle_{s\in S}$ a coalgebra structure making it a complete Hopf algebra. The coproduct is determined by making each element X_s primitive, i.e. $\Delta(X_s) = 1\widehat{\otimes}X_s + X_s\widehat{\otimes}1$. The counit is given by $\varepsilon(X_s) = 0$. We know that there are isomorphism $\widehat{kF}(S) \cong k\langle\langle X_s\rangle\rangle_{s\in S} \cong \widehat{U}L(S)$. Explicitely the isomorphism

$$\varphi : \widehat{kF}(S) \xrightarrow{\cong} k\langle\langle X_s\rangle\rangle_{s\in S}$$

is given by $\varphi(s) = exp(X_s)$ for $s \in S$. The isomorphism

$$\theta : \widehat{U}L(S) \xrightarrow{\cong} k\langle\langle X_s\rangle\rangle_{s\in S}$$

is given by $\theta(S) = X_s$ for $s \in S$.

Definition 6.8.1. The complete Hopf algebra $k\langle\langle X_s\rangle\rangle_{s\in S}$ is called the **free** complete Hopf algebra on the set S. In general a complete Hopf algebra H will be called **free** if it is isomorphic to the free complete Hopf algebra on a set S.

The free complete Hopf algebras play an important role in $\mathbf{Hopf}^{\text{comp}}$.

Proposition 6.8.1. *A complete Hopf algebra is projective if and only if it is isomorphic to a free complete Hopf algebra $k\langle\langle X_s\rangle\rangle_{s\in S}$ on S, for some set S.*

Theorem 6.8.1. *Any complete Hopf algebra H is isomorphic to the quotient of a free complete Hopf algebra P by a closed Hopf ideal (cf. Definition 6.6.2). The free complete Hopf algebras are the projective objects in $\mathbf{Hopf}^{comp}$.*

For the proof of both these results see [Qui69, Appendix A, Proposition 2.22 and Corollary 2.23].

The existence and plentifulness of free complete Hopf algebras allow us prove results about complete Hopf algebras using generators and relations. In particular one can prove the following result ([Qui69, Appendix A, Proposition 2.24]).

Theorem 6.8.2. *The category $\mathbf{Hopf}^{comp}$ of complete Hopf algebras is complete and cocomplete.*

Remark 6.8.1. Giving explicit descriptions of limits in categories of Hopf algebras seems quite difficult. As late as 2010 work was done by Agore (see [Ag011]) giving descriptions of all limits in the category of "ordinary" (i.e. not filtered)

Hopf algebras, with no cocompleteness assumption. The author does not know if the methods of [Ago11] could be applied to give explicit descriptions of limits in $\mathbf{Hopf}^{\mathrm{comp}}$. At least the issue of cocommutativity should not pose too many problems. Indeed, the category of "ordinary" cocommutative Hopf algebras is coreflective in the category of Hopf algebras. Given an "ordinary" Hopf algebra H, the coreflector is the operation of taking the subset $H^{cocomm} \subseteq H$ of cocommutative elements (i.e. elements $x \in H$ such that $\Delta x = \tau \Delta x$ where τ is the "switch" map). Then H^{cocomm} is in fact a Hopf subalgebra and the association $H \mapsto H^{cocomm}$ is right adjoint to the inclusion of the category of cocommutative Hopf algebras into the category of Hopf algebras.

6.9 Model structure

We now have almost all the properties required in order to use Theorem 5.4.2. The candidate for the functor $G : \mathbf{Hopf}^{\mathrm{comp}} \to \mathrm{Set}$ is the functor $\mathcal{P}$ of primitive elements (or, equivalently, the functor $\mathcal{G}$ of group-like elements). Here we think og $\mathcal{P}$ as a Set-valued functor, disregarding the fact that it factors through the category of Lie algebras. Thus we make the following candidate definitions.

Definition 6.9.1. Let $r \geq 0$ and let $f : H \to H'$ be a morphism in $(\mathbf{sHopf}^{\mathrm{comp}})_r$.

(i) Call f a *weak equivalence* if $G(f)$ is a weak equivalence in $\mathbf{sSet}_r$.
(ii) Call f a *fibration* if $G(f)$ is a fibration in $\mathbf{sSet}_r$.
(iii) Call f a *cofibration* if it has the LLP with respect to all acyclic fibrations.

Remark 6.9.1. In order to prove that the above definitions yield a model structure on the category $\mathbf{sHopf}_r^{\mathrm{comp}}$ of r-reduced simplicial complete Hopf algebras, we must show that $G = \mathcal{P}$ preserves all filtered colimits. However, it appears that it does not! The reason being the inherently infinitary definition of *complete* Hopf algebras . See also Remark 7.3.2 for the corresponding situation in the unproblematic case of simplicial Lie algebras.

As Remark 6.9.1 shows, we cannot directly us Theorem 5.4.2 to put a model structure on $\mathbf{sHopf}_r^{\mathrm{comp}}$. However, the proof of Theorem 5.4.2 only uses the assumption that G preserves filtered colimits once, namely in the proof of the factorization axiom (Lemma 5.3.2) using the small object argument. Thus, we can salvage the conclusion of Theorem 5.4.2 applied to $\mathcal{C} = \mathbf{sHopf}_r^{\mathrm{comp}}$ if we can find some other way of proving the factorization axiom. The solution is to use the fact (stated in Theorem 6.8.1) that $\mathbf{Hopf}^{\mathrm{comp}}$ has enough projectives, to make a factorization anyway.

Lemma 6.9.1. *Any morphism $f : H \to H'$ of complete Hopf algebras may be factored as $f = pi$ where i is a cofibration and p is an acyclic fibration.*

The result as it appears in [Qui67, Part II, chap 4, Proposition 3] relies on the theory of effective epimorphisms.

Theorem 6.9.1. *Definition 6.9.1 puts a model structure on* $\mathbf{sHopf}_r^{comp}$*, the category of r-reduced simplicial complete Hopf algebras.*

6.10 Connections with simplicial groups

With the model structure now placed on $\mathbf{sHopf}_r^{\mathrm{comp}}$ we can discuss the functors $\mathcal{G}$ and $\widehat{\mathbb{Q}}$ in the context of Section 2.6. First, we need a result proved by Quillen using results by Curtis ([Cur65]).

Proposition 6.10.1. *If G is a reduced almost free simplicial group then the unit* $\eta : G \to \mathcal{G}\widehat{\mathbb{Q}}G$ *is a weak equivalence in* $\mathbf{sGrp}_0^{\mathbb{Q}}$

See [Qui69, Part I, Theorem 3.4] for a proof.

Proposition 6.10.2. *The pair* $(\widehat{\mathbb{Q}}, \mathcal{G})$ *of adjoint functors form a Quillen equivalence.*

$$\mathbf{sGrp}_r^{\mathbb{Q}} \underset{\mathcal{G}}{\overset{\widehat{\mathbb{Q}}}{\rightleftarrows}} \mathbf{sHopf}_r^{comp}$$

In particular they induce an adjoint equivalence of categories $\mathrm{Ho}(\mathbf{sGrp}_r^{\mathbb{Q}}) \cong \mathrm{Ho}(\mathbf{sHopf}_r^{comp})$.

Proof. By Lemma 6.7.1 the two functors are adjoint. We know that cofibrant simplicial groups in $\mathbf{sGrp}_r^{\mathbb{Q}}$ are the r-reduced almost free simplicial groups, and that all complete Hopf algebras are fibrant. Suppose G is a cofibrant r-reduced simplicial group and H is a complete Hopf algebra. Given a weak equivalence $f : \widehat{\mathbb{Q}}G \xrightarrow{\sim} H$ in $\mathbf{sHopf}_r^{\mathrm{comp}}$, the adjoint of f is the composition

$$G \xrightarrow{\eta} \mathcal{G}\widehat{\mathbb{Q}}G \xrightarrow{\mathcal{G}f} \mathcal{G}H.$$

This is a weak equivalence since η is (by Proposition 6.10.1) and since $\mathcal{G}$ preserves weak equivalences, as is seen by the natural isomorphism $\mathcal{G} \cong \P$.
Conversely, suppose $g : G \xrightarrow{\sim} \mathcal{G}H$ is a weak equivalence. We must show that the adjoint $\widehat{\mathbb{Q}}G \to H$ is a weak equivalence in $\mathbf{sHopf}_r^{\mathrm{comp}}$. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{P}\widehat{\mathbb{Q}}G & \xrightarrow{\mathcal{P}(\widehat{\mathbb{Q}}(g))} & \mathcal{P}\widehat{\mathbb{Q}}\mathcal{G}H & \xrightarrow{\mathcal{P}(\varepsilon)} & \mathcal{P}H \\
{\cong}\downarrow{\exp} & & {\cong}\downarrow{\exp} & & {\cong}\downarrow{\exp} \\
\mathcal{G}\widehat{\mathbb{Q}}G & \xrightarrow{\mathcal{G}\widehat{\mathbb{Q}}(g)} & \mathcal{G}\widehat{\mathbb{Q}}\mathcal{G}H & \xrightarrow{\mathcal{G}(\varepsilon)} & \mathcal{G}H \\
{\eta}\uparrow{\sim} & & & \nearrow_{g}^{\sim} & \\
G & & & &
\end{array}$$

By the 2-out-of-3 property for weak equivalences in $\mathbf{sGrp}_r^{\mathbb{Q}}$ we see that the adjoint $\widehat{\mathbb{Q}}G \to H$ is indeed a weak equivalence in $\mathbf{sHopf}_r^{\mathrm{comp}}$. Thus, $(\widehat{\mathbb{Q}}, \mathcal{G})$ form a Quillen equivalence. The second claim follows from Theorem 2.6.2. □

Chapter 7

Simplicial Lie Algebras

The purpose of this chapter is to introduce the category of Lie algebras and then the category of simplicial Lie algebras. We will not come close to a full development of the exciting theory of Lie algebras. For a very elementary introduction (with focus on classification of finite dimensional Lie algebras over $\mathbb{C}$) see [EW06]. For a more general and sophisticated approach see e.g. Serre's book [Ser92] about Lie algebras and Lie groups.
We give a proof that **Lie** is a complete and cocomplete category and use this to put a model structure on **sLie**.

7.1 The category of Lie algebras

Let k be a field of characteristic zero. As usual the focus is on the case $k = \mathbb{Q}$.

Definition 7.1.1. A **Lie algebra** over k is a k-vector space $\mathfrak{g}$ equipped with a linear map $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, called the **(Lie) bracket**, which is *antisymmetric* (i.e. $[x, y] = -[y, x]$) and satisfies the *Jacobi identity*: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$. A k-linear map $f : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras is a **Lie homomorphism** if $f([x, y]) = [fx, fy]$ for all $x, y \in \mathfrak{g}$.

The category of Lie algebras and Lie homomorphisms is denoted **Lie**. We will be considering some aspects of the category **Lie**.

Remark 7.1.1. A few elementary observations.

(i) Lie algebras are not assumed to be associative, and generally they won't be.
(ii) Since k has characteristic zero the antisymmetry condition is equivalent to the **alternating** condition on the bracket, i.e. $[x, x] = 0$ for all $x \in \mathfrak{g}$.
(iii) Using the bilinearity and antisymmetry we can rewrite the Jacobi identity as $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ which shows that the operator $[x, -]$ satisfies the **Leibniz** identity.

(iv) Lie algebras are defined in terms of sets and finitary operations satisfying certain identities, thus they form a variety (in the sense of universal algebra). See for instance [Ber15, chap. 9, sec. 7]. From this it follows that **Lie** is both complete and cocomplete. However, we shall not use this machinery but instead prove these results "by hand".

(v) The set of all Lie homomorphisms $\mathrm{Hom}_{\mathbf{Lie}}(\mathfrak{g}, \mathfrak{h})$ between two Lie algebras, $\mathfrak{g}$ and $\mathfrak{h}$, is naturally a k-vector space with point-wise addition and scalar multiplication. The zero map is the neutral element.

If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ is a subspace then we say that $\mathfrak{h}$ is a **Lie subalgebra** if $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$. If furthermore $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$ and $y \in \mathfrak{g}$ then $\mathfrak{h}$ is called a **(Lie) ideal** of $\mathfrak{g}$.

Lemma 7.1.1. *If $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie ideal, then the Lie bracket on $\mathfrak{g}$ induces a Lie bracket on the quotient space $\mathfrak{g}/\mathfrak{h}$. Any subspace $V \subseteq \mathfrak{g}$ is contained in a smallest Lie ideal $\langle V \rangle \subseteq \mathfrak{g}$.*

The proof is straightforward (see e.g. [EW06, chap. 2]).

Example 7.1.1. The trivial vector space 0 admits a Lie algebra structure with trivial bracket. This is both an initial and terminal object in **Lie**.

Example 7.1.2. For every field k there is a unique 1-dimensional Lie algebra $\mathfrak{g}$, i.e. Lie algebra with 1-dimensional underlying vector space. The bracket must in this case be 0 since all elements $x, y \in \mathfrak{g}$ are linearly dependent i.e. there is some $\lambda \in k$ such that $y = \lambda x$. Then $[x, y] = [x, \lambda x] = \lambda[x, x] = 0$.

Example 7.1.3. If A is an associative algebra over k then consider A_{Lie} the Lie algebra which has A as underlying k-vector space. The bracket is given by the commutator $[x, y] = xy - yx$. The associativity of the product on A ensures the Jacobi identity holds. A map $f : A \to A'$ of associative k-algebras preserves the commutator and so induces a map $f_{Lie} : A_{Lie} \to A'_{Lie}$. This association is functorial.

Proposition 7.1.1. *The category* **Lie** *of Lie algebras is complete.*

Proof. Given a family $\{\mathfrak{g}_i\}_{i\in I}$ (where I is some set) of Lie algebras with brackets $[,]_i$, the product vector space $\prod_i \mathfrak{g}_i$ may be equipped with a Lie bracket defined as $[(a_i)_{i\in I}, (b_i)_{i\in I}] = ([a_i, b_i]_i)_{i\in I}$. The projection maps are Lie homomorphisms and it is clear that the required universal property is satisfied. Similarly, equalizers may be computed in **Vect** and then equipped with brackets: given $f, f' : \mathfrak{g} \to \mathfrak{h}$ parallel Lie homomorphisms, consider the subspace $\mathrm{Ker}(f - f') \subseteq \mathfrak{g}$. Then $\mathrm{Ker}(f - f')$ is stable under the bracket in $\mathfrak{g}$ and so inherits the Lie algebra structure from $\mathfrak{g}$. It is clear that this is an equalizer in **Lie**. The proposition now follows from the general fact that categories with arbitrary (small) products and with equalizers have all (small) limits. □

The functor $(-)_{Lie} : \mathbf{Alg} \to \mathbf{Lie}$ from Example 7.1.3 is forgetting the product structure on A and only remembering the commutator. As is often the case with "forgetful" functors, there is a left adjoint $U : \mathbf{Lie} \to \mathbf{Alg}$ called the **universal enveloping algebra**. Given a Lie algebra $\mathfrak{g}$, a universal enveloping algebra $U\mathfrak{g}$ together with a Lie homomorphism $i : \mathfrak{g} \to U\mathfrak{g}$ must have the following universal property. For any associative algebra A with a Lie homomorphism $\varphi : \mathfrak{g} \to A_{Lie}$ there exists a unique map of algebras $\widehat{\varphi} : Ug \to A$ such that $\widehat{\varphi} \circ i = \varphi$. Note that if $U\mathfrak{g}$ exists it is unique up to isomorphism of associative k-algebras.

Proposition 7.1.2. *Every Lie algebra $\mathfrak{g}$ has a universal enveloping algebra $U\mathfrak{g}$.*

Proof. Recall the construction of the tensor algebra on a vector space V; this is the free associative algebra on V and may be constructed as follows. Let $T^nV = V^{\otimes n}$ be the n-fold tensor product (over k) of V with itself, where $V^{\otimes 0} = k$. Then

$$TV = \bigoplus T^nV.$$

This is indeed a free associative algebra on V, i.e. has the correct universal property. Let $I \subseteq T\mathfrak{g}$ be the ideal generated by $[x, y] - x \otimes y - y \otimes x$ for $x, y \in \mathfrak{g}$. Then $T\mathfrak{g}/I$ together with the obvious map $i : \mathfrak{g} \to T\mathfrak{g} \to T\mathfrak{g}/I$ is a universal enveloping algebra for $\mathfrak{g}$. □

7.2 The Poincaré-Birkhoff-Witt theorem

The Poincaré-Birkhoff-Witt theorem comes in several somewhat different, yet equivalent, versions. One important corollary is a representation theorem, showing that any Lie algebra is in fact a Lie subalgebra of an associative algebra.

Let a Lie algebra $\mathfrak{g}$ over k be given. Choose a basis $\{e_\alpha\}_{\alpha \in A}$ for $\mathfrak{g}$ as a k-vector space. We may assume the index set A is well-ordered. If $I = (\alpha_1, \ldots, \alpha_n)$ is a sequence of indices then we denote by e_I the product $e_{\alpha_1} \cdots e_{\alpha_n}$ in $U\mathfrak{g}$. (This is a slight abuse of notation, this should really read $e_I = i(e_{\alpha_1}) \cdots i(e_{\alpha_n})$). Call a sequence *I increasing* if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Define $e_\varnothing = 1$.

Theorem 7.2.1. *(Poincaré-Birkhoff-Witt) The set $\{e_I \in U\mathfrak{g} : I$ an increasing sequence $\}$ forms a basis for $U\mathfrak{g}$ as a k-vector space.*

The proofs (at least those found by the author) are quite involved. See [Ser92, chap. 3, Theorem 4.3] or, for dg versions of the theorem see [FHT01, chap. 21, Theorem 21.1] [Qui69, Appendix B]. If $I = (\alpha)$ then $e_I = i(e_\alpha) \in U\mathfrak{g}$ is the image of $e_\alpha \in \mathfrak{g}$ under the map $i : \mathfrak{g} \to U\mathfrak{g}$. By the theorem, these images are all linearly independent in $U\mathfrak{g}$. From this observation we get the following corollary.

Corollary 7.2.1. *The map $i : \mathfrak{g} \to U\mathfrak{g}$ is injective. Thus $\mathfrak{g}$ is a Lie subalgebra of $(U\mathfrak{g})_{Lie}$.*

7.3 Free Lie algebras and colimits

The Poincaré-Birkhoff-Witt theorem allows us to construct *free* Lie algebras.

Proposition 7.3.1. *The forgetful functor* **Lie** $\to$ **Vect** *which forgets the bracket has a left adjoint* $\mathbb{L}$: **Vect** $\to$ **Lie** *called the* ***free*** *Lie algebra functor.*

Proof. Given a vector space V, define $\mathbb{L}(V)$ to be the Lie subalgebra of $(TV)_{Lie}$ (cf. Example 7.1.3) generated by $V = T^1V \subseteq TV$. Suppose $\mathfrak{g}$ is some Lie algebra and $f : V \to \mathfrak{g}$ is a k-linear map. Composing with $i : \mathfrak{g} \to U\mathfrak{g}$ gives a map $V \overset{i\circ f}{\to} U\mathfrak{g}$ into an associative algebra. This extends to a unique algebra map $\bar{f} : TV \to U\mathfrak{g}$. By Corollary 7.2.1, $\mathfrak{g}$ is a Lie subalgebra of $U\mathfrak{g}$, thus since $\mathbb{L}(V)$ is generated by V the map $\bar{f} : TV \to U\mathfrak{g}$ restricts to a map $\mathbb{L}(V) \to \mathfrak{g}$ which is a Lie homomorphism since $\bar{f}$ is. □

In particular, if X is a set then the k-vector space kX generated by X gives rise to $\mathbb{L}(kX)$. We shall call $\mathbb{L}(kX)$ the free Lie algebra generated by X and denote it by $\mathbb{L}(X)$, as in the following corollary.

Corollary 7.3.1. *The forgetful functor* **Lie** $\to$ Set *which forgets the Lie algebra and vector space structure has a left adjoint* $\mathbb{L}$: Set $\to$ **Lie** *(also) called the* ***free*** *Lie algebra functor.*

Proof. One constructs $\mathbb{L}$ by composing with the *free k-vector space functor* Set $\to$ **Vect**. □

Proposition 7.3.2. *The category* **Lie** *of Lie algebras is cocomplete.*

Proof. We show that **Lie** has coequalizers and small coproducts. Suppose $f_1, f_2 : \mathfrak{g} \to \mathfrak{h}$ are parallel Lie homomorphisms. The coequalizer is given as follows. The image $\mathrm{Im}(f_1 - f_2)$ is a Lie subalgebra. Let J be the smallest Lie ideal containing $\mathrm{Im}(f_1 - f_2)$ (cf. Lemma 7.1.1). Then $\mathfrak{h}/J$ is a coequalizer. This follows from the homomorphism theorems for Lie algebras (see [EW06, chap. 2]).
We now construct coproducts. Let $\{\mathfrak{g}_i\}_{i\in I}$ be a set-indexed family of Lie algebras. The Lie bracket in $\mathfrak{g}_i$ is denoted $[,]_i$. Consider the coproduct (i.e. direct sum) of the underlying k-vector spaces $\bigoplus_{i\in I}\mathfrak{g}_i$ with the linear inclusions $in_j : \mathfrak{g}_j \to \bigoplus_{i\in I}\mathfrak{g}_i$. Let L denote the free Lie algebra on the vector space $\bigoplus_{i\in I}\mathfrak{g}_i$ with bracket $[,]_L$. Let $\iota_j : \mathbb{L}(\mathfrak{g}_j) \to L$ be the map $\mathbb{L}(in_j)$. Let J be the Lie ideal generated by the relations

$$\iota_j([g,g']_j) = [\iota_j(g), \iota_j(g')]_L \qquad \text{for all } j \in I \text{ and all } g, g' \in \mathfrak{g}_j.$$

Let $\pi : L \to L/J$ be the quotient Lie homomorphism. The maps $\pi \circ \iota_j$ are Lie homomorphisms for all $j \in I$ and the pair $(L/J, \{\pi \circ \iota_j\}_{j\in I})$ is universal among such pairs, i.e. a coproduct of the family $\{\mathfrak{g}_i\}_{i\in I}$.
Since categories with coequalizers and (small) coproducts are cocomplete ([Mac71, chap. 5, sec. 2]), we are done. □

Remark 7.3.1. The construction of coproducts in **Lie** is unfortunately not very explicit. The author does not know of a more explicit description. The situation is reminiscent of the coproduct (also known as *free product*) in **Grp**. To see how things get complicated one can try to describe the coproduct of two 1-dimensional Lie algebras $\mathfrak{g}_1 \cong k \cong \mathfrak{g}_2$ (cf. Example 7.1.2). Let $[,]_i$ denote the bracket in $\mathfrak{g}_i$ $(i = 1, 2)$ and $[,]$ the bracket in the coproduct $\mathfrak{g}_1 \amalg \mathfrak{g}_2$. If $x \in \mathfrak{g}_1$ and $y \in g_2$ then there is no obvious reduction of $[x, y]$ nor of $[x, [x, y]]$ or $[[x, y], y]$, etc..

We shall now prove that the forgetful functor G : **Lie** $\to$ Set preserves filtered colimits. This relies on the following lemma which says that filtered colimits commute with finite limits in Set. A *finite* category is a category with only finitely many objects and morphisms.

Theorem 7.3.1. *Let I be a small filtered category and J a finite category. If $X : I \times J \to$ Set is a diagram in Set then the canonical map*

$$\mathrm{colim}_I \mathrm{lim}_J X(i, j) \longrightarrow \mathrm{lim}_J \mathrm{colim}_I X(i, j)$$

is a bijection.

The proof of this fundamental result is not hard, however it does rely on a precise description of filtered colimits in Set and so will be omitted. See [Mac71, chap. 9, Theorem 2.1] for the details.

Corollary 7.3.2. *The forgetful functor U :* **Lie** $\to$ Set *preserves filtered colimits.*

Proof. The crucial observation is that Lie algebras are defined as sets equipped with certain finitary operations on them. In more detail, a Lie algebra $\mathfrak{g}$ over k may be described as a tuple $(\mathfrak{g}, [\ ,\ ], +, -, 0, \{\lambda\}_{\lambda \in k})$ where $\mathfrak{g}$ is a set and the other entries are functions

$$\mathfrak{g}^2 \xrightarrow{[\ ,\ ],+} \mathfrak{g} \qquad \mathfrak{g} \xrightarrow{-,\lambda} \mathfrak{g} \qquad \mathfrak{g}^0 \cong * \xrightarrow{0} \mathfrak{g}$$

between various products of $\mathfrak{g}$. The vector space axioms and Lie algebra axioms can all be described in terms of commutative diagrams. For example, the Jacobi identity may be "coded" as the requirement that the following diagram is commutative.

$$\begin{array}{ccccc}
\mathfrak{g}^3 & \xrightarrow{(id,\sigma,\sigma^2)} & \mathfrak{g}^3 \times \mathfrak{g}^3 \times \mathfrak{g}^3 & \xrightarrow{(id,[,])^3} & \mathfrak{g}^2 \times \mathfrak{g}^2 \times \mathfrak{g}^2 \\
\downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle [\ ,\ ]\times[\ ,\ ]\times[\ ,\ ]} \\
\mathfrak{g} & \xleftarrow[+]{} & \mathfrak{g} \times \mathfrak{g} & \xleftarrow[(id,+)]{} & \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}
\end{array}$$

Here $0 : \mathfrak{g}^3 \to \mathfrak{g}$ is really the composite $\mathfrak{g}^3 \to * \xrightarrow{0} \mathfrak{g}$. The map $\sigma : \mathfrak{g}^3 \to \mathfrak{g}^3$ is the cyclic permutation $(x, y, z) \longmapsto (y, z, x)$. Suppose we are given a diagram $I \to \mathbf{Lie}$ where I is filtered. Composing with G we get a diagram

$$X : I \to \mathbf{Lie} \xrightarrow{G} \mathrm{Set}$$

in Set where each set $X(i)$ may be equipped with a Lie algebra structure in the form of the functions $f(i) : X(i)^n \to X(i)^m$ for various n and m. These functions $f(i) : X(i)^n \to X(i)^m$ actually define natural transformations between the diagrams given by the term-wise m-fold and n-fold products. By naturality, each transformation $f : X^n \to X^m$ induces a map $\mathrm{colim}_I(f)$ on the colimits of the diagrams

$$\mathrm{colim}_I(X^n) \xrightarrow{\mathrm{colim}_I(f)} \mathrm{colim}_I(X^m).$$

Since taking k-fold products is a finite limit, Theorem 7.3.1 implies that the natural map $\mathrm{colim}_I(X^k) \to \mathrm{colim}_I(X)^k$ is a bijection. Thus we can uniquely define a map $\tilde{f} : \mathrm{colim}_I(X)^n \to \mathrm{colim}_I(X)^m$ by the following diagram

$$\begin{array}{ccc} \mathrm{colim}_I(X^n) & \xrightarrow{\mathrm{colim}_I(f)} & \mathrm{colim}_I(X^m) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{colim}_I(X)^n & \overset{}{\underset{\tilde{f}}{\dashrightarrow}} & \mathrm{colim}_I(X)^m \end{array}$$

By naturality, the association $f \mapsto \tilde{f}$ is functorial i.e. given two composable operations $f : X^n \to X^m$ and $g : X^m \to X^k$ then $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$ and $\widetilde{id} = id$. Thus, the operations $\tilde{f}$ satisfy the same relations as the maps f do. For example, taking $f = + : X^2 \to X$ the associativity relation $+ \circ (id, +) = + \circ (+, id)$ gives $\tilde{+} \circ (id, \tilde{+}) = \tilde{+} \circ (\tilde{+}, id)$. So $\tilde{+} : \mathrm{colim}_I(X)^2 \to \mathrm{colim}_I(X)$ is associative. Finally, the universal maps $in_i : X(i) \to \mathrm{colim}_I X$ into the colimit are Lie homomorphisms, since each of the maps in the diagram is a Lie homomorphism. Therefore the $\mathrm{colim}_I X$ together with the maps $\{in_i\}_{i \in I}$ is the colimit in **Lie**. □

Remark 7.3.2. As the proof of Corollary 7.3.2 shows, the fact that G preserves filtered colimits is directly related to the "finitary definition" of Lie algebras. The same arguments therefore apply to many other common "algebraic" categories $\mathcal{C}$ with forgetful functors $G : \mathcal{C} \to \mathrm{Set}$. For instance, the forgetful functor from groups or rings also preserves filtered colimits. However, in the category of *complete* Hopf algebras this strategy fails. A complete Hopf algebra H is required to satisfy the *in*finitary condition $H = \varprojlim_n H/F_nH$ where $\{F_n\}_{n \in \omega}$ is a specified filtration. This issue required us to take a slightly different approach when proving the model category axioms for the category of simplicial complete Hopf algebras, cf. Remark 6.9.1.

7.4 Model structure

In Section 5.3 we discussed a general method of putting a (simplicial) model structure on $(s\mathcal{C})_r$ where $\mathcal{C}$ is a sufficiently "algebraic" category. We shall use this method to put a model structure on **sLie**, the category of simplicial objects in **Lie**. Let us state the proposed model structure. We shall refer to the forgetful functor $G : \mathbf{sLie} \to \mathbf{sSet}$ (the prolongation of the forgetful functor $\mathbf{Lie} \to \mathrm{Set}$) as well as its left adjoint $\mathbb{L} : \mathbf{sSet} \to \mathbf{sLie}$.

Definition 7.4.1. Let $r \geq 0$ and let $f : \mathfrak{g} \to \mathfrak{h}$ be a morphism $(\mathbf{sLie})_r$.

(i) Call f a *weak equivalence* if $G(f)$ is a weak equivalence in $\mathbf{sSet}_r$.
(ii) Call f a *fibration* if $G(f)$ is a fibration in $\mathbf{sSet}_r$.
(iii) Call f a *cofibration* if it has the LLP with respect to all acyclic fibrations.

Theorem 7.4.1. *The model structure as proposed in Definition 7.4.1 equips* **Lie** *with a model category structure.*

Proof. We verify the conditions of Definition 5.4.3. We have shown in Proposition 7.1.1 and Proposition 7.3.2 that **Lie** is complete and cocomplete. As pointed out in Example 7.1.1, **Lie** is a pointed category. The functors G and F of Definition 5.4.3 may be taken to be U and $\mathbb{L}$ respectively. By Corollary 7.3.2 G commutes with filtered colimits. □

7.5 Connection with $\mathbf{sHopf}^{\mathbf{comp}}$

In this section we will prove that the adjoint functors $\widehat{\mathcal{U}} \dashv \mathcal{P}$ form a Quillen equivalence (cf. Section 2.6). As a result they induce an equivalence of categories between $\mathrm{Ho}(\mathbf{sHopf}_r^{\mathrm{comp}})$ and $\mathrm{Ho}(\mathbf{sLie}_r)$.
First, we need a result proved by Quillen using results by Curtis ([Cur65]).

Proposition 7.5.1. *If $\mathfrak{g}$ is a reduced almost free simplicial Lie algebra then the unit $\eta : \mathfrak{g} \to \mathcal{P}\widehat{\mathcal{U}}\mathfrak{g}$ is a weak equivalence in* $\mathbf{sLie}_r$.

See [Qui69, Part I, Theorem 3.5] for a proof.

Proposition 7.5.2. *The pair $(\widehat{\mathcal{U}}, \mathcal{P})$ of adjoint functors form a Quillen equivalence.*

$$\mathbf{sHopf}_r^{comp} \underset{\widehat{\mathcal{U}}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathbf{sLie}_r$$

In particular they induce an adjoint equivalence of categories $\mathrm{Ho}(\mathbf{sLie}_r) \cong \mathrm{Ho}(\mathbf{sHopf}_r^{comp})$.

Proof. By Lemma 6.7.1 the two functors are adjoint with $\widehat{U}$ left adjoint. We know that cofibrant simplicial Lie algebras in $\mathbf{sLie}_r$ are the r-reduced almost free simplicial Lie algebras. Also, all r-reduced complete Hopf algebras are fibrant. Suppose $\mathfrak{g}$ is a cofibrant r-reduced simplicial Lie algebra and H is an r-reduced complete Hopf algebra. Given a weak equivalence $f : \widehat{U}\mathfrak{g} \underset{\sim}{\to} H$ in $\mathbf{sHopf}_r^{\mathrm{comp}}$, the adjoint of f is the composition

$$\mathfrak{g} \xrightarrow{\ \eta\ } \mathcal{P}\widehat{U}\mathfrak{g} \xrightarrow{\ \mathcal{P}f\ } \mathcal{P}H.$$

This is a weak equivalence since η is (by Proposition 7.5.1) and since $\mathcal{P}$ preserves weak equivalences, by definition.

Conversely, suppose $g : \mathfrak{g} \underset{\sim}{\to} \mathcal{P}H$ is a weak equivalence in $\mathbf{sLie}_r$. We must show that the adjoint $\widehat{U}\mathfrak{g} \to H$ is a weak equivalence in $\mathbf{sHopf}_r^{\mathrm{comp}}$, i.e. that the induced map $\mathcal{P}\widehat{U}\mathfrak{g} \to \mathcal{P}H$ is a weak equivalence in $\mathbf{sSet}_r^{\mathrm{Q}}$. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{P}\widehat{U}\mathfrak{g} & \xrightarrow{\ \mathcal{P}\widehat{U}g\ } & \mathcal{P}\widehat{U}\mathcal{P}H & \xrightarrow{\ \mathcal{P}\varepsilon\ } & \mathcal{P}H \\
{\scriptstyle \eta}\uparrow{\scriptstyle \sim} & & & \nearrow{\scriptstyle g,\ \sim} & \\
\mathfrak{g} & & & &
\end{array}$$

By the 2-out-of-3 property for weak equivalences in $\mathbf{sSet}_r^{\mathrm{Q}}$ we see that the adjoint $\widehat{U}\mathfrak{g} \to H$ is indeed a weak equivalence in $\mathbf{sHopf}_r^{\mathrm{comp}}$. Thus, $(\widehat{\mathrm{Q}}, \mathcal{G})$ form a Quillen equivalence. The second claim follows from Theorem 2.6.2. □

Chapter 8

Differential Graded Lie Algebras

In this chapter we introduce the functor N taking a simplicial Lie algebra to a differential graded Lie algebra. We then put a model structure on $\mathbf{dgLie}_r$ the category of r-reduced dg Lie algebras and show that N is part of a Quillen equivalence between $\mathbf{sLie}_r$ and $\mathbf{dgLie}_r$.

8.1 The normalization functor

Let **sAb** be the category of simplicial Abelian groups. A morphism in **sAb** is a morphism of simplicial sets where each component is a group homomorphism. Let **dgAb** be the category of differential graded Abelian groups where the grading is nonnegative, i.e. objects of **dgAb** are sequences $A = \{A_n\}_{n\geq 0}$ of Abelian groups, equipped with a differential $d : A \to A$ of degree -1 such that $d^2 = 0$. Differential graded Abelian groups will also be referred to their more common name: **chain complexes** (over the ring $\mathbb{Z}$) . We now define the functor

$$N : \mathbf{sAb} \longrightarrow \mathbf{dgAb}$$

of *normalized chains*. On objects we have

$$(NA)_n = \bigcap_{0\leq i<n} \ker(d_i) \subseteq A_n$$

using all of the face maps d_i except the top face, d_n. The map d_n is used to construct the differential

$$NA_n \xrightarrow{(-1)^n d_n} NA_{n-1}.$$

Note that the simplicial identity $d_{n-1}d_n = d_{n-1}d_{n-1}$ ensures that $d \circ d = 0$ i.e. that we have created a chain complex.

Remark 8.1.1. This definition makes sense even for possibly *non*-Abelian groups. The result is then of course not a chain complex.

Given a morphism $\varphi : A \to B$ of simplicial Abelian groups we get a morphism of chain complexes $N(\varphi)$ by restricting φ_n to NA_n. This works since φ_n is a group homomorphism and so commutativity of the diagram

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & B_n \\ \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} \\ A_{n-1} & \xrightarrow{\varphi_{n-1}} & B_{n-1} \end{array}$$

for $0 \leq i < n$ ensures that φ_n restricts to a homomorphism

$$A_n \supseteq \bigcap_{0 \leq i < n} \ker(d_i) \longrightarrow \bigcap_{0 \leq i < n} \ker(d_i) \subseteq B_n.$$

Furthermore the commutativity of the diagram for $i = n$ ensures that $N(\varphi)$ is a morphism of chain complexes. Clearly $N(\varphi \circ \psi) = N(\varphi) \circ N(\psi)$ and $N(\mathrm{id}) = \mathrm{id}$. Thus N is a functor.

8.2 What is being normalized?

Why do we call N the *normalized* chains functor? This comes from the comparison with two other ways of turning simplicial Abelian groups into chain complexes. First there is the usual construction, used in simplicial homology, taking A to the chain complex (A, ∂) with degree n equal to A_n and with differential

$$\partial = \sum_{i=0}^{n} (-1)^i d_i.$$

This is called the *Moore complex* of A. Let $DA \subseteq A$ be the subcomplex which is generated by the degenerate simplices. Given $x = s_j(x')$ in DA_n and computing the differential

$$\begin{aligned} \partial(x) &= \sum_{i=0}^{n} (-1)^i d_i s_j(x') \\ &= \sum_{i<j} (-1)^i d_i s_j(x') + (-1)^j d_j s_j(x') + (-1)^{j+1} d_{j+1} s_j(x') + \sum_{i>j+1} (-1)^i d_i s_j(x') \\ &= s_{j-1}\left(\sum_{i<j} (-1)^i d_i(x')\right) + s_j\left(\sum_{i>j+1} (-1)^i d_{i-1}(x')\right) \end{aligned}$$

shows that $\partial(x)$ is in DA_{n-1}, the subgroup generated by the degeneracies in A_{n-1}. Thus ∂ induces a map

$$\partial : A_n / DA_n \longrightarrow A_{n-1} / DA_{n-1}$$

making A/DA into a chain complex.

Proposition 8.2.1. *NA is isomorphic to A/DA as chain complexes.*

Proof. The isomorphism will be the composition

$$NA \xrightarrow{i} A \xrightarrow{p} A/DA.$$

It suffices to show that this is a bijection in each level. For this we use the following filtration

$$A_n \supset N_0 A_n \supset N_1 A_n \supset \cdots \supset N_{n-1}A_n = (NA)_n$$

and the resulting diagram

$$A_n \twoheadrightarrow A_n/D_0A_n \twoheadrightarrow A_n/D_1A_n \twoheadrightarrow \cdots \twoheadrightarrow A_n/D_{n-1}A_n = (A/DA)_n$$

where $N_jA_n = \bigcap_{0\le i\le j} \mathrm{Ker}(d_i)$ and D_jA_n is the subgroup generated by all degeneracies coming from s_i's for $i \le j$.
We will show, by induction on j, that

$$N_jA_n \longrightarrow A_n \longrightarrow A_n/D_jA_n,$$

is an isomorphism. Call this map φ_j^n. If φ_j^n is an isomorphism for all j then taking $j = n-1$ yields the required isomorphism.
Suppose $j = 0$. To show surjectivity suppose $[x] \in A_n/D_0A_n$. Then $[x]$ may be represented by $x - s_0d_0x$. Using the simplicial identity $d_0s_0 = id$ and using the fact that d_0 is a homomorphism we have $d_0(x - s_0d_0x) = 0$. Thus $x - s_0d_0x \in N_0A_n$ maps via φ_0^n to $[x]$.
To show injectivity suppose $\varphi_0^n(x) = 0$. Thus $x \in D_0A_n$. At level 0 we have a particularly simple description of D_0A_n. It contains elements of the form $s_0(y)$ for $y \in A_{n-1}$ (the simplicity of this description fails for $j > 0$). Thus $x = s_0(y)$. Since $x \in N_0A_n$ we have

$$0 = d_0x = d_0s_0y = y$$

so $x = 0$. Thus φ_0^n is an isomorphism for all n.
Now suppose we have shown that $\varphi_k^m : N_kA_m \longrightarrow A_m/D_kA_m$ is an isomorphism for all $k < j$ and all m. Consider the commutative diagram

$$\begin{array}{ccc} N_{j-1}A_n & \xrightarrow[\cong]{\varphi_{j-1}^n} & A_n/D_{j-1}A_n \\ \uparrow & & \downarrow \\ N_jA_n & \xrightarrow{\varphi_j^n} & A_n/D_j(A_n) \end{array}$$

where the vertical arrows are the obvious maps. Suppose $\alpha \in A_n/D_j(A_n)$ then since φ_{j-1}^n is an isomorphism there is some representative $x \in \alpha$ such that $x \in$

$N_{j-1}A_n$. Then $x - s_j d_j x$ is in $N_j A_n$ and is also a representative. Thus $\varphi_j^n(x - s_j d_j x) = \alpha$, so φ_j^n is surjective.
To show injectivity we first make some remarks. The map $s_j : A_{n-1} \to A_n$ takes $N_{j-1}A_{n-1}$ into $N_{j-1}A_n$ and takes $D_{j-1}A_{n-1}$ to $D_{j-1}A_n$, thus inducing a map $\widetilde{s_j} : A_{n-1}/D_{j-1}A_{n-1} \longrightarrow A_n/D_{j-1}A_n$. Consider the commutative diagram

$$\begin{array}{ccc}
N_{j-1}A_{n-1} & \xrightarrow[\cong]{\varphi_{j-1}^{n-1}} & A_{n-1}/D_{j-1}A_{n-1} \\
\downarrow{\scriptstyle s_j} & & \downarrow{\scriptstyle \widetilde{s_j}} \\
N_{j-1}A_n & \xrightarrow[\cong]{\varphi_{j-1}^{n}} & A_n/D_{j-1}A_n \\
\uparrow & & \downarrow \\
N_j A_n & \xrightarrow[\varphi_j^n]{} & A_n/D_j A_n
\end{array}$$

where the bottom square is the same as in the previous commuting square. Now the right-hand column of this diagram is short exact; the induced map $\widetilde{s_j}$ is injective since $s_j : A_{n-1} \to A_n$ is a section of d_j (and since the top square is commutative with horizontal bijections). The rest of the exactness conditions are clear.
We can now prove injectivity by a diagram chase: if $\varphi_j^n(x) = 0$ then $\varphi_{j-1}^n(x)$ is in the kernel of the map $A_n/D_{j-1}A_n \twoheadrightarrow A_n/D_j A_n$ hence in the image of $\widetilde{s_j}$, and so x is of the form $s_j y$ for $y \in N_{j-1}A_{n-1}$. Since $d_j x = 0$ we have

$$0 = d_j x = d_j s_j y = y$$

and so $x = 0$. Thus φ_j^n is an isomorphism. By induction we are done. □

8.3 The Normalization Theorem

We have seen that NA is isomorphic to A/DA. We can also relate NA directly to A, thinking of A as a chain complex.

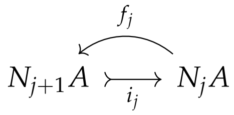
Proposition 8.3.1. *The normalized chain complex NA is chain homotopic to A.*

Proof. As in the proof that NA is isomorphic to A/DA we split the problem of finding the chain homotopy into parts indexed by $j = 0, \ldots, n-1$.
For a fixed $j \geq 0$ we define

$$N_j A_n = \begin{cases} \bigcap_{i=0}^{j} \mathrm{Ker}(d_i) & \text{if } n \geq j+2 \\ NA_n & \text{if } n \leq j+1 \end{cases}$$

The differential $\partial = \sum_{i=0}^{n}(-1)^i d_i$ restricts to give a differential on the subgroups $N_j A_n$. For $n \leq j+1$ this is clear. For $n \geq j+2$ one can check that $\partial(x)$ is killed

by any d_k for $k \leq j$, using the simplicial identity $d_k d_i = d_{i-1} d_k$ and also using the homomorphism property of d_k. Thus $N_j A$ is a subcomplex of A.
By definition $N_{j+1}A \subseteq N_j A$. Call the inclusion map i_j. This is a map of chain complexes. Define $f_j : N_j A \to N_{j+1} A$ in the other direction,



by

$$f_j(x) = \begin{cases} x - s_{j+1}d_{j+1} & \text{if } n \geq j+2 \\ x & \text{if } n \leq j+1 \end{cases} .$$

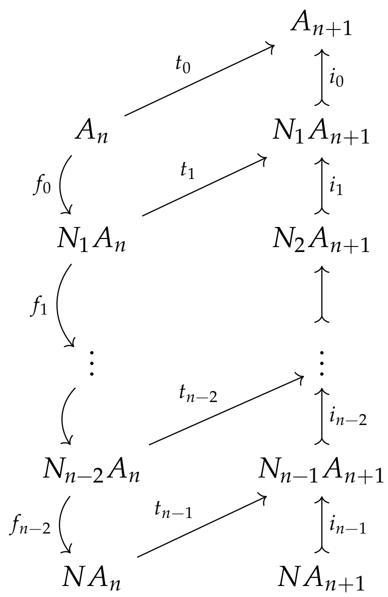
Note that $f_j \circ i_j = id_{N_{j+1}A}$.
We thus have maps $f = f_{n-2} \cdots f_0 : A_n \longrightarrow NA_n$ and $i = i_0 \cdots i_{n-1} : NA_n \longrightarrow A_n$ for each n. These maps satisfy $f \circ i = id_{NA_n}$.
It remains to show that $i \circ f \simeq id_A$. Let $t_j : N_j A_n \to N_j An + 1$ be defined by

$$t_j(x) = \begin{cases} (-1)^j s_{j+1} & \text{if } n \geq j+1 \\ 0 & \text{if } n \leq j \end{cases} .$$

Then t_j defines a chain homotopy between $id_{N_j A}$ and $i_j \circ f_j$. Now we paste the maps t_j together. Fix n and define $T_n : A_n \to A_{n+1}$ to be

$$T_n = i_0 \cdots i_{n-2} t_{n-1} f_{n-2} \cdots f_0 + i_0 \cdots i_{n-3} t_{n-2} f_{n-3} \cdots f_0 + i_0 t_1 f_0 + t_0$$

To help parse T_n it may be helpful to keep this diagram in mind :



Then one can show that $(T_n)_{n \geq 0}$ defines a chain homotopy between $f \circ i$ and id_A. This completes the proof. □

Proposition 8.3.1 in particular implies that to compute the homology of a given chain complex it suffices to compute the homology of the much small normalized complex!

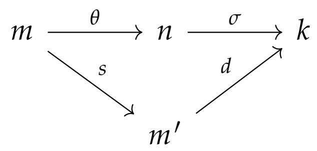
8.4 The Dold-Kan Correspondence

It would seem that we have thrown away a lot of information when passing to the normalized chains. Indeed we have "forgotten" all simplices that are killed by lower face maps. However, we now show that any simplicial Abelian group can be reconstructed from its normalized chains.

Given a simplicial Abelian group A, consider the simplicial Abelian group $N^{-1}A$ defined as follows:

$$N^{-1}A_n = \bigoplus_{n \twoheadrightarrow k} NA_k$$

where the (finite) sum is taken over all surjections $n \twoheadrightarrow k$ in Δ. Given a map $\theta : m \to n$ in Δ we define $\theta^* = N^{-1}A(\theta)$ as follows; on summand $(A_k, \sigma : n \twoheadrightarrow k)$ first form the epi-mono factorization of $\sigma \circ \theta$



and then define θ^* as the composition

$$NA_k \xrightarrow{d^*} NA_{m'} \xrightarrow{in_s} N^{-1}A_m.$$

The Dold-Kan theorem says that N is in fact an equivalence of categories. The inverse may be explicitly described as follows. Let $N^{-1} : \mathbf{dgAb} \to \mathbf{sAb}$ be the following functor. For A a dg Abelian group we let

$$N^{-1}A_n = \bigoplus_{n \twoheadrightarrow k} A_k$$

The i'th face $d_i : N^{-1}A_n \to N^{-1}A_{n-1}$ is given as follows: on the component corresponding to $\sigma : n \twoheadrightarrow m$ first make the epi-mono factorization of σd^i say $\sigma d^i = ds$. Let ι_s be the inclusion of the component corresponding to s. Now one the component corresponding to σ we let $d_i(x) = \iota_s(x)$ if $d = id$. If $d = d^j$ for some $j < m$ then set $d_i = 0$. Finally, if $d = d^m$ then we set $d_i(x) = (-1)^m \iota_s \partial(x)$. The i'th degeneracy $s_i : N^{-1}A_n \to N^{-1}A_{n+1}$ is defined on component corresponding to σ by

$$s_i(x) = \iota_{\sigma s^i}(x).$$

Given a map $f : A \to B$ of dg Abelian groups the map $N^{-1}\varphi : N^{-1}A \to N^{-1}B$ is defined at degree n, on the summand A_k corresponding to $\sigma : n \twoheadrightarrow k$, by the composite

$$A_k \xrightarrow{f_k} B_k \xrightarrow{\iota_\sigma} \bigoplus_{n \twoheadrightarrow k} B_k$$

where ι_σ denotes the summand inclusion. The following proposition is now a matter of checking the simplicial identities hold. We will not write out the details.

Lemma 8.4.1. *The definitions just given define a functor* $N^{-1} : \mathbf{dgAb} \to \mathbf{sAb}$.

Theorem 8.4.1. *(Dold-Kan) The functors* N *and* N^{-1} *form an equivalence of categories between* **sAb** *and* **dgAb**.

See [GJ99, chap. 2, Corollary 2.3] or [May67, chap. 22, Theorem 22.4] for proofs. One can show in fact show that N^{-1} is left adjoint to N.

8.5 Differential graded Lie algebras

We shall work over $k = \mathbb{Q}$ the field of rationals.

Definition 8.5.1. A **graded Lie algebra** is a graded vector space $L = \{L_p\}_{p \geq 0}$ equipped with a linear map $[,] : L \otimes L \to L$ of degree 0 (called the **Lie bracket)** which satisfies *graded antisymmetry*, i.e.

$$[x, y] = -(-1)^{pq}[y, x]$$

for homogeneous elements $x, y \in L$ of degree p and q respectively. We also require a *graded Jacobi identity*

$$(-1)^{pr}[x, [y, z]] + (-1)^{qp}[y, [z, x]] + (-1)^{rq}[z, [x, y]] = 0$$

for homogeneous elements $x, y, z \in L$ of degree p, q, r respectively.

Definition 8.5.2. A **morphism** $f : L \to L'$ of graded Lie algebras is a linear map of degree 0 which preserves the bracket, i.e. $f[x, y] = [fx, fy]$.

Example 8.5.1. If X is a simply connected topological space then the graded $\mathbb{Q}$-vector space $\pi_*(\Omega X) \otimes \mathbb{Q}$ equipped with the bracket operation

$$[\alpha, \beta] = (-1)^{\deg(\alpha)+1}\partial_*([\partial_*^{-1}\alpha, \partial_*^{-1}\beta]_W)$$

(where $\partial_* : \pi_* X \xrightarrow{\cong} \pi_{*-1}(\Omega X)$ is the connecting homomorphism (induced by the fibration $\Omega X \to \mathcal{P}X \to X$) and $[,]_W$ is the Whitehead product) is a graded Lie algebra, called the **rational homotopy Lie algebra** of X. See [FHT01, chap 21(d)] for a proof.

Example 8.5.2. Given a graded algebra A and a graded Lie algebra L the tensor product (as graded vector spaces) $A \otimes L$ may be given the structure of a graded Lie algebra with the bracket operation given by

$$[a \otimes x, a' \otimes x'] = (-1)^{\deg(a)\deg(x)} aa' \otimes [x, x']$$

for homogeneous elements $a, a' \in A$ and homogenous elements $x, x' \in L$.

Let $(L, [\ ])$ be a graded Lie algebra. A **derivation** of L of degree $p \in \mathbb{Z}$ is a linear map $\theta \in \mathrm{Hom}(L, L)$ such that

$$\theta[x, y] = [\theta x, y] + (-1)^{p|x|}[x, \theta y]$$

for all homogeneous elements $x, y \in L$. The set of derivations of degree p is closed under addition and thus forms a vector space denoted $\mathrm{Der}_p(L)$. Then $\mathrm{Der}(L) = \{\mathrm{Der}_p\}_{p \in \mathbb{Z}}$ is the graded space of all derivations on L. A **differential** ∂ on L is a derivation of degree -1 such that $\partial \circ \partial = 0$.

Definition 8.5.3. A **differential graded Lie algebra** is a triple $(L, [\ ,\ ], \partial)$ where $(L, [\ ,\ ])$ is a graded Lie algebra and ∂ is a differential.

Definition 8.5.4. A **morphism** $f : (L, \partial) \to (L', \partial')$ of differential graded Lie algebras is a morphism of differential graded vector spaces which is also a morphism of graded Lie algebras.

The category of differential graded Lie algebras and morphisms between them is denoted **dgLie**.
The next proposition is, while not difficult, important for rational homotopy theory. The proposition shows that *homology* defines a functor **dgLie** $\to$ **gLie** i.e. takes values in graded Lie algebras. Quillen's original question (or at least one version of it) concerned how to lift the construction of the rational homotopy Lie algebra (cf. Example 8.5.1) along the homology functor.

Proposition 8.5.1. *If L is a dg Lie algebra then the homology $H(L)$ is a graded Lie algebra with the Lie bracket defined on representatives.*

Proof. The homology $H(L)$ is certainly a graded vector space. If $a \in Z_p(L)$ and $b \in Z_q(L)$ are cycles then $[a, b] \in L_{p+q}$ is also a cycle since ∂ is a derivation. Also the bracket respects the boundary relation, i.e. if $\partial\tilde{a} \in B_p(L)$ is a boundary then for any $x \in Z_q(L)$ the derivation property shows that

$$[\partial a, x] = \partial[\tilde{a}, x] - (-1)^{(p+1)q}[\tilde{a}, \partial x] = \partial[\tilde{a}, x]$$

So $[,] : L \otimes L \to L$ induces a well-defined linear map $[,] : H(L) \otimes H(L) \to H(L)$. The Jacobi identity and antisymmetry are then directly verified. □

See Section 7.1 for the corresponding non-graded definitions of simplicial Lie algebras.

8.6 The Eilenberg-Mac Lane map

Given simplicial Abelian groups K and L define their product $K \otimes L$ to be the simplicial Abelian group $(K \times L)_n = K_n \otimes_{\mathbb{Z}} L_n$, the dimension-wise tensor product. In general, given simplicial k-modules V and W we let $V \otimes W$ denote the dimension-wise tensor product, $(V \otimes W)_n = V_n \otimes W_n$.
For clarity let $M : \mathbf{sAb} \to \mathbf{dgAb}$ be the Moore complex construction, i.e. $M(A)$ is the chain complex with differential $\partial = \sum_i (-1)^i d_i$.

Definition 8.6.1. Let p and q be nonnegative integers. A (p,q)-shuffel is a permutation $(\mu_1, \cdots, \mu_p, \nu_1, \cdots, \nu_q)$ of $\{0, 1, \cdots, p+q+1\}$ such that $\mu_1 < \mu_2 < \cdots < \mu_p$ and $\nu_1 < \nu_2 < \cdots < \nu_q$.

Following May ([May67, chap 29, Definition 29.7]) we shall refer to the *Eilenberg-Mac Lane* map $EM : M(V) \otimes M(W) \longrightarrow M(V \otimes W)$, the map of chain complexes defined for $x \in V_q$ and $y \in W_p$ as

$$EM(x \otimes y) = \sum_{(\mu,\nu)} (-1)^{\varepsilon(\mu)} s_{\nu_q} \cdots s_{\nu_1}(x) \otimes s_{\mu_p} \cdots s_{\mu_1}(y)$$

where the sum ranges over all (p,q)-shuffles (μ, ν) and where $\varepsilon(\mu) := \sum_{i=1}^{p} \mu_i - (i-1)$.

Remark 8.6.1. The *Alexander-Whitney map* $AW : M(V \otimes W) \longrightarrow M(A) \otimes M(W)$ going in the other direction is in fact a homotopy-inverse to EM (see [May67, chap. 29, Corollary 29.10]).

We will need the following properties of the Eilenberg-Mac Lane map.

Proposition 8.6.1. *The Eilenberg-Mac Lane map satisfies the following properties. Here x has degree p and y has degree q.*

(i) $EM(x, EM(y,z)) = EM(EM(x,y),z)$ *(Associativity)*
(i) $d(EM(x,y)) = EM(dx,y) + (-1)^p EM(x,dy)$
(i) $\tau EM(x,y) = (-1)^{pq} EM(y,x)$ *where τ is the twist map* $\tau(x \otimes y) = y \otimes x$.
(i) If $x \in NV_p$ and $y \in NW_q$ then $EM(x,y) \in N(V \otimes W)_{p+q}$ and the chain map

$$(NV) \otimes (NW) \xrightarrow{N(EM)} N(V \otimes W) \qquad x \otimes y \mapsto EM(x,y).$$

is a chain homotopy equivalence.

For proofs of these claims we refer to [EML53, chap. 5].

8.7 Reshuffling a simplicial Lie algebra

Viewing Abelian groups as $\mathbb{Z}$-modules suggests generalizing the functor N to other ground rings. Working over a field k, N becomes a functor

$$N : \mathbf{sVect} \longrightarrow \mathbf{dgVect}$$

between the category of simplicial vector spaces and differential graded vector spaces.

Now suppose $\mathfrak{g}$ is a simplicial Lie algebra. Forgetting the Lie structure form the Moore complex $M(\mathfrak{g})$ giving a chain complex of k-vector spaces. As before $N(\mathfrak{g})$ is a subcomplex (recall the differential in $N(\mathfrak{g})$ is given a sign $(-1)^n$) of $M(\mathfrak{g})$. Now the composition

$$M(\mathfrak{g}) \otimes M(\mathfrak{g}) \xrightarrow{EM} M(\mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{\beta} M(\mathfrak{g})$$

defines a binary operation, denoted $[[\ ,\ ]]$, on $M(\mathfrak{g})$. By the normalization property of Proposition 8.6.1 $[[\ ,\ ]]$ restricts to a binary operation (also denoted $[[\ ,\ ]]$) on $N(\mathfrak{g})$. Furthermore, by the rest of Proposition 8.6.1 we have the following proposition.

Proposition 8.7.1. *$N(\mathfrak{g})$ with $[[\ ,\ ]]$ is a differential graded Lie algebra.*

So the normalized chains functor may be viewed as a functor $N : \mathbf{sLie} \to \mathbf{dgLie}$ between simplicial Lie algebras and dg Lie algebras.

8.8 The left adjoint N^*

We have seen that $N : \mathbf{sVect} \to \mathbf{dgVect}$ is an equivalence of categories with inverse given by $N^{-1} : \mathbf{dgVect} \to \mathbf{sVect}$. We have also seen that N restricts to a functor on the subcategory of simplicial Lie algebras. We now show that the restricted functor N admits a left adjoint.

First we note that the forgetful functor $U : \mathbf{Lie} \to \mathbf{Vect}$ admits a left adjoint $\mathbb{L} : \mathbf{Vect} \to \mathbf{Lie}$ called the *free Lie algebra* functor. Given a vector space V the universal property of $\mathbb{L}V$ is the following: there is a natural bijection between Lie algebra morphisms $\mathbb{L}V \to \mathfrak{g}$ and vector space morphisms $V \to U(\mathfrak{g})$. One way of constructing $\mathbb{L}(V)$ is by taking the Lie sub-algebra generated by V inside the free tensor algebra $T(V)$ on V (cf. the non-graded construction in Section 7.3).

Given a dg Lie algebra $\mathfrak{m}$ we can forget the Lie bracket and view $\mathfrak{m}$ as a dg vector space. Applying N^{-1} yields a simplicial vector space $N^{-1}\mathfrak{m}$. Then as dg vector spaces there is a natural isomorphism $\mathfrak{m} \cong NN^{-1}\mathfrak{m}$ coming from the Dold-Kan correspondence. Viewing $N^{-1}\mathfrak{m}$ as a dg vector space (via the Moore complex construction) we have

$$\mathfrak{m} \cong NN^{-1}\mathfrak{m} \subseteq N^{-1}\mathfrak{m}.$$

This gives a function $f_{N^{-1}} : \mathfrak{m} \to N^{-1}\mathfrak{m}$.

Proposition 8.8.1. *The functor* N : **sLie** $\to$ **dgLie** *has a left adjoint* N^* : **dgLie** $\to$ **sLie**.

Proof. Let $\mathfrak{m}$ be a dg Lie algebra. Forget the Lie structure and apply N^{-1} to get a simplicial vector space $N^{-1}\mathfrak{m}$. Apply the free Lie algebra functor in each dimension, giving a simplicial Lie algebra $\mathbb{L}N^{-1}\mathfrak{m}$. Given a simplicial Lie algebra $\mathfrak{g}$ and a map $\varphi : \mathfrak{m} \to N\mathfrak{g}$ apply N^{-1} to get

$$N^{-1}\mathfrak{m} \xrightarrow{N^{-1}(\varphi)} N^{-1}N\mathfrak{g} \cong \mathfrak{g}$$

where the isomorphism is an isomorphism of simplicial vector spaces, coming from the Dold-Kan correspondence. The composite map extends uniquely to a map of simplicial Lie algebras $\theta : \mathbb{L}N^{-1}\mathfrak{m} \longrightarrow \mathfrak{g}$. This map θ thus satisfies that $\theta(f_{N^{-1}}(x)) = \varphi(x)$ for $x \in \mathfrak{m}$. We now force the extension θ to exist only when φ was in fact a Lie homomorphism. Define

$$N^*\mathfrak{m} = \mathbb{L}N^{-1}\mathfrak{m}/I$$

where I is the simplicial ideal of $\mathbb{L}N^{-1}\mathfrak{m}$ generated by $[[f_{N^{-1}}(x), f_{N^{-1}}(y)]] - f_{N^{-1}}[x,y]$ for x, y homogeneous elements of $\mathfrak{m}$. The first bracket is the one given by the shuffle construction on $N^{-1}\mathfrak{m}$ and the second bracket is from $\mathfrak{m}$. Now in order for the map

$$\theta : \mathbb{L}N^{-1}\mathfrak{m} \longrightarrow \mathfrak{g}$$

to induce a map on N^*, the ideal I must be contained in the kernel of the map. Thus we require

$$\theta(f_{N^{-1}}[x,y]) = \theta([[f_{N^{-1}}x, f_{N^{-1}}y]]).$$

Using the relation $\theta(f_{N^{-1}}(x)) = \varphi(x)$ for $x \in \mathfrak{m}$ and that θ is a Lie homomorphism this equality is equivalent to the equality

$$\varphi[x,y] = [[\varphi x, \varphi y]].$$

Thus there is an induced map of simplicial Lie algebras $\theta : N^*\mathfrak{m} \to \mathfrak{g}$ if and only if $\varphi : \mathfrak{m} \to N\mathfrak{g}$ is a dg Lie algebra homomorphism.
The bijection $\mathrm{Hom}_{\mathbf{dgLie}}(\mathfrak{m}, N\mathfrak{g}) \cong \mathrm{Hom}_{\mathbf{sLie}}(N^*\mathfrak{m}, \mathfrak{g})$ thus constructed is clearly natural and so the proof is complete. □

8.9 Preservations of Freeness

We shall now show that the left adjoint N^* constructed above carries free dg Lie algebras (defined below) to almost free simplicial Lie algebras. For the definition of almost free see Appendix A.4.

Definition 8.9.1. A dg Lie algebra (L, d) is called **free** if the underlying graded Lie algebra L is isomorphic to a free graded Lie algebra $\mathbb{L}^g V$ for some graded vector space V.

Warning! 1. Note that we only say (L, d) is free if L is freely generated in the category of graded Lie algebras. Specifically $(\mathbb{L}^g V, d)$ need not be freely generated in the category of *differential* graded Lie algebras.

If V is a graded vector space then we shall denote by $V^{\leq k}$ the *k-truncated* graded vector space with

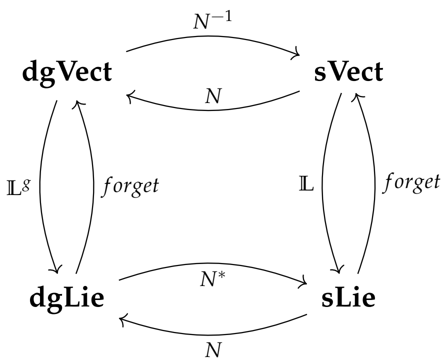
$$(V^{\leq k})_j = \begin{cases} 0 & \text{if } j > k \\ V_j & \text{if } j \leq k \end{cases} .$$

Lemma 8.9.1. *The graded Lie subalgebra E of $\mathbb{L}^g V$ generated by V_i for $i \leq k$ is isomorphic to $\mathbb{L}^g V^{\leq k}$, the free graded Lie algebra on the k-truncation of V.*

Proof. We check the universal property. Let $\varphi : V^{\leq k} \to L$ be a linear map of graded vector spaces with L a graded Lie algebra. Extend φ to a linear map $\widehat{\varphi} : V \to L$ by setting $\widehat{\varphi} = 0$ on degrees higher than k. This then extends to a map of graded Lie algebras $\widehat{\widehat{\varphi}} : \mathbb{L}^g V \to L$ which restricts to the required map of graded Lie algebras $\widetilde{\varphi} : E \to L$ extending φ. This extension is unique since E is generated, as a graded Lie algebra, by $V^{\leq k}$ and the action of $\widetilde{\varphi}$ is determined (equal to φ) on this generating set. □

Lemma 8.9.2. *The two functors $N^*\mathbb{L}^g$ and $\mathbb{L}N^{-1}$ are isomorphic.*

Proof. This will follow from the uniqueness of left adjoints. The situation is depicted in the following diagram of adjoints.



The functors $N \circ forget$ and $forget \circ N$ from **sLie** to **dgVect** are *equal*. To see this note that all Lie brackets are forgotten so it remains to check that the differential is the same, but in both cases we use the differential given from the normalized chains construction. Now $N^*\mathbb{L}^g$ and $\mathbb{L}N^{-1}$ are left adjoints to $N \circ forget = forget \circ N$, thus $N^*\mathbb{L}^g \cong \mathbb{L}N^{-1}$ □

Recall the chain complex models of the sphere and disk: The chain complex $S(k-1)$ has single generator y_{k-1} in degree $k-1$, with $d(y_{k-1}) = 0$. The chain complex $D(k)$ has a generator x_k of degree k and a generator y_{k-1} of degree $k-1$ and has $dx_k = y_{k-1}$ (thus $dy_{k-1} = 0$). In the following the map $S(k-1) \hookrightarrow D(k)$ denotes the inclusion, which is a chain map.
Recall also (a possible version of) the simplicial set models of the sphere and disk: The "simplicial k-sphere" A is given by the quotient

$$A = \Delta^{k-1}/\partial\Delta^{k-1}.$$

The "simplicial k-disk" B is given by the quotient

$$B = \Delta^k/\Lambda^k_0$$

where $\Lambda^k_0 \subseteq \Delta^k$ is 0'th k-horn, generated by all faces $d_j(\iota_k)$ except the 0'th face $d_0(\iota_k)$. The inclusion $\Delta^{k-1} \hookrightarrow \Delta^k$ into the last face (the one which is missing in the 0'th horn) induces a map $A \to B$, which corresponds to the inclusion of the sphere as the boundary of the disk.
Let $\overline{\mathbb{Q}}$: $\mathbf{sSet}_* \to \mathbf{sVect}$ be the functor from pointed simplicial sets to simplicial vector spaces which generates the free $\mathbb{Q}$-space on the given simplicial set, with the zero vector identified with the base-point.

Lemma 8.9.3. *The maps $N^{-1}S(k-1) \to N^{-1}D(k-1)$ and $\overline{\mathbb{Q}}A \to \overline{\mathbb{Q}}B$ in* **sVect** *are isomorphic.*

Proof. The map $f : N^{-1}S(k-1) \to \overline{\mathbb{Q}}A$ is defined as follows. For degrees $n < k-1$ both vector spaces are trivial. In degree $k-1$ there is a unique summand of $N^{-1}S(k-1)$ corresponding to $id_k : k \twoheadrightarrow k$, namely $\mathbb{Q}y_{k-1}$. Likewise A_{k-1} is a two-point set with base-point the class $[\partial\Delta^{k-1}_{k-1}]$ and the other element given by $id_{k-1} : k-1 \to k-1$. Thus $(\overline{\mathbb{Q}}A)_{k-1} = \mathbb{Q}\{id_{k-1}\}$, so f_{k-1} is defined on basis vectors by $y_{k-1} \mapsto id_{k-1}$. For higher degree's $n > k-1$ the maps $N^{-1}S(k-1)_n \xrightarrow{f_n} \overline{\mathbb{Q}}A_n$ are determined by the unique representation of surjections in Δ; given a surjection $\sigma : n \twoheadrightarrow m$ with $m < n$ there exists unique factorization

$$\sigma = s^{j_1}s^{j_2}\cdots s^{j_{n-m}}$$

where $0 \le j_{n-m} < \cdots < j_1 \le m$. The elements of A_n for $n > k-1$ are precisely the degeneracies of the map id_{k-1} together with the base-point $[\partial\Delta^{k-1}_n]$ (which is the set of all the degeneracies of elements of Δ^{k-1}_l for $l < k-1$). So

$$f_n : N^{-1}S(k-1)_n \to (\overline{\mathbb{Q}}A)_n$$

is determined by mapping a basis element y_{k-1} corresponding to $s^{j_1}s^{j_2}\cdots s^{j_{n-m}}$: $n \twoheadrightarrow m$ to the basis element $s^{j_1}s^{j_2}\cdots s^{j_{n-m}} \in A_n$. The resulting map $f = (f_n)$: $N^{-1}S(k-1) \to \overline{\mathbb{Q}}A$ is a simplicial map, which clearly is an isomorphism.

The simplicial vector space isomorphism $g : N^{-1}D(k) \to \overline{\mathbb{Q}}B$ is defined similarly. From the definitions of f and g it may be checked that the diagram

$$\begin{CD} N^{-1}S(k-1) @>>> N^{-1}D(k) \\ @V{f}V{\cong}V @V{\cong}V{g}V \\ \overline{\mathbb{Q}}A @>>> \overline{\mathbb{Q}}B \end{CD}$$

is commutative, thus an isomorphism in the arrow category. □

We can now state and prove the preservation property of the left adjoint functor N^* constructed earlier.

Proposition 8.9.1. *The functor N^* :* **dgLie** $\to$ **sLie** *takes free dg Lie algebras to almost free simplicial Lie algebras.*

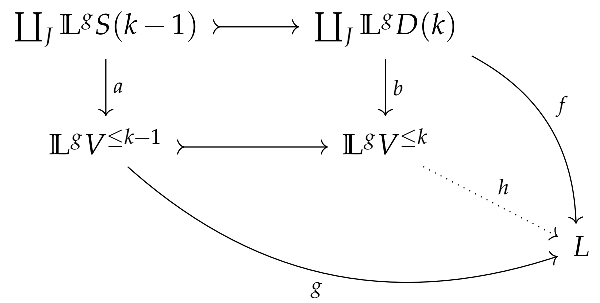
Proof. Let $\mathfrak{m}$ be a free dg Lie algebra, say $\mathfrak{m} = (\mathbb{L}^g V, d)$, where V is a graded vector space. Define $\mathfrak{m}^{(k)}$ to be the differential graded Lie sub-algebra generated by V_i for $i \leq k$. As a graded Lie algebra $m^{(k)}$ is just the graded Lie sub-algebra generated by V_i (for $i \leq k$) since the differential preserves the subset $\bigcup_{i\leq k} V_i$. So $m^{(k)} = (\mathbb{L}^g V^{\leq k}, d)$ by the Lemma 8.9.1.
Now pick a basis $\{e_j : j \in J\}$ for V_k. We claim that the following diagram in **dgLie** is a pushout diagram,

$$\begin{CD} \coprod_J \mathbb{L}^g S(k-1) @>>> \coprod_J \mathbb{L}^g D(k) \\ @V{}V{a}V @V{}V{b}V \\ \mathfrak{m}^{(k-1)} @>>> \mathfrak{m}^{(k)} \end{CD}$$

where a restricted to the j'th component is determined by $a(y_{k-1}) = d(e_j)$ and where b restricted to the j'th component is determined by $b(x_k) = e_j$ (thus $b(y_{k-1}) = de_j$). The coproducts in the top row are taken in the category **dgLie**.
To prove that the diagram is a pushout suppose given a dg Lie algebra L and dg Lie homomorphisms f and g such that the diagram

$$\begin{CD} \coprod_J \mathbb{L}^g S(k-1) @>>> \coprod_J \mathbb{L}^g D(k) \\ @V{}V{a}V @V{}V{b}V \\ \mathfrak{m}^{(k-1)} @>>> \mathfrak{m}^{(k)} \end{CD} \qquad f: \coprod_J \mathbb{L}^g D(k) \to L,\quad g: \mathfrak{m}^{(k-1)} \to L,\quad h: \mathfrak{m}^{(k)} \dashrightarrow L$$

(without h) commutes. We show that there exists a unique dg Lie homomorphism h making the diagram commute. Forgetting the differentials we may consider the diagram in **gLie**. It then looks as follows



Now we define h by defining it on $V^{\leq k}$. Elements $v \in V^{\leq k}$ of degree $< k$ are mapped to L via g. The basis vector $e_j \in V_k$ is mapped to $f(x_j)$. This determines a *unique* lift $h : \mathbb{L}^g V^{\leq k} \longrightarrow L$. By construction it makes the complete diagram commute. Furthermore since $\mathfrak{m}^{(k)}$ is generated, as a graded Lie algebra, by V_i, $i \leq k$ and since h respects the differential of this generating set (by construction, since f and g are dg Lie homomorphisms) h is actually a dg Lie homomorphism. Thus the original diagram is indeed a pushout in **dgLie**. Thinking in terms of cell complexes we have shown that $\mathfrak{m}^{(k)}$ is obtained from $\mathfrak{m}^{(k-1)}$ by attaching the "cells" e_j for $j \in J$.

Now apply the left adjoint functor N^* to the pushout diagram. Since N^* is left adjoint it preserves all colimits, and so we get the following pushout diagram in **sLie**:

$$\begin{array}{ccc} \coprod_J N^*\mathbb{L}^g S(k-1) & \longrightarrow & \coprod_J N^*\mathbb{L}^g D(k) \\ \big\downarrow a & & \big\downarrow b \\ N^*\mathfrak{m}^{(k-1)} & \longrightarrow & N^*\mathfrak{m}^{(k)} \end{array}$$

where we can no longer assume the horizontal arrows are monic. Our aim is now to show that the maps $N^*\mathbb{L}^g S(k-1) \to N^*\mathbb{L}^g D(k)$ are almost free maps. Since almost free maps are closed under pushouts and sequential compositions this will show that $0 \to \mathrm{colim}_k N^*\mathfrak{m}^{(k)}$ is almost free. Finally since N^* is left adjoint we have $\mathrm{colim}_k N^* m^{(k)} \cong N^* \mathrm{colim}_k m^{(k)} = N^*\mathfrak{m}$. So we will have shown that $0 \to N^*\mathfrak{m}$ is almost free, as claimed.

By Lemma 8.9.2 $N^*\mathbb{L}^g$ is isomorphic to $\mathbb{L}N^{-1}$ so it suffices to show that

$$\mathbb{L}N^{-1}S(k-1) \longrightarrow \mathbb{L}N^{-1}D(k)$$

is almost free. By Lemma 8.9.3 lemma the map $N^{-1}S(k-1) \to N^{-1}D(k)$ is isomorphic to the map $\overline{\mathbb{Q}}A \to \overline{\mathbb{Q}}B$, so it suffices to show that

$$\mathbb{L}\overline{\mathbb{Q}}A \xrightarrow{f} \mathbb{L}\overline{\mathbb{Q}}B$$

is almost free. Let $X_q \subseteq \mathbb{L}\overline{\mathbb{Q}}B_q$ be the subset of B_q consisting of those elements not in the image of the injection $A_q \to B_q$. Then evidently $\mathbb{L}\overline{\mathbb{Q}}A_q \amalg \mathbb{L}\overline{\mathbb{Q}}X_q \cong \mathbb{L}\overline{\mathbb{Q}}B_q$ and this isomorphism may be chosen such that it acts as f on the $\mathbb{L}\overline{\mathbb{Q}}A_q$-component. Finally given a codegeneracy $s^j : q+1 \to q$ in Δ the map $s_j : \mathbb{L}\overline{\mathbb{Q}}B_q \to \mathbb{L}\overline{\mathbb{Q}}B_{q+1}$ maps X_q into X_{q-1} since this is true already at the level of $B_q \to B_{q+1}$. Thus $f : \mathbb{L}\overline{\mathbb{Q}}A \to \mathbb{L}\overline{\mathbb{Q}}B$ is almost free and the proof is complete.

□

In [Qui69] Quillen uses Proposition 8.9.1 to prove that the unit of the adjunction $N^* \dashv N$ is a weak equivalence when the component is a free dg Lie algebra. Specifically he proves the following result.

Theorem 8.9.1. *Let $\mathfrak{m}$ be a free reduced dg Lie algebra, then the unit map $\eta : \mathfrak{m} \to NN^*\mathfrak{m}$ is a quasi-isomorphism, i.e. induces an isomorphism on homology.*

See [Qui69, Part I, Theorem 4.6] for a proof.

8.10 Model Structure on reduced dg Lie algebras

In this section we put a model structure on the category of reduced dg Lie algebras such that the weak equivalences are maps inducing isomorphism on homology, commonly called *quasi-isomorphisms*. Recall that in this thesis differential graded objects are always assumed to be non-negatively graded. See Definition 8.5.3 for the definition and basic properties of differential graded Lie algebras.

Definition 8.10.1. A dg Lie algebra L is said to be r-reduced if $L_i = 0$ for $i \leq r$. We let $\mathbf{dgLie}_r$ denote the full subcategory of **dgLie** consisting of r-reduced dg Lie algebras.

(i) A map f in $\mathbf{dgLie}_r$ is called a *weak equivalence* if it induces isomorphisms on homology.
(ii) A map f in $\mathbf{dgLie}_r$ is called a *fibration* if it is surjective in degrees $> r+1$.
(iii) A map f in $\mathbf{dgLie}_r$ is called a *cofibration* if it has the LLP with respect to all acyclic fibrations.

Lemma 8.10.1. *The category $\mathbf{dgLie}_r$ is complete and cocomplete.*

Theorem 8.10.1. *The category $\mathbf{dgLie}_r$ with the proposed model structure of Definition 8.10.1 is a model category.*

In order to prove this theorem it is helpful to set up a couple of tools. Let $\mathbb{L}^g$: **gVect** $\to$ **gLie** be the free graded Lie algebra functor, left adjoint to the forgetful functor. Its construction is very similar to the construction of the free Lie algebra in Section 7.3. See [FHT01, chap. 21(c)] for the construction.

Definition 8.10.2. Let $S(q)$ be the differential graded vector space (over $\mathbb{Q}$) which has one generator σ_q in degree q and is 0 everywhere else. Note that the differential is zero. This is the analogue of the q-sphere in **dgVect**. Taking the free dg Lie algebra $\mathbb{L}^g S(q)$ yields the analogue of the q-sphere in **dgLie**. Similarly let $D(q)$ be the dg vector space which has one generator τ_q in degree q and one generator σ_{q-1} in degree $q-1$. The differential is determined by the requirement $d(\tau_q) = \sigma_{q-1}$. Again, taking the free dg Lie algebra $\mathbb{L}^g D(q)$ yields the analogue of the q-disc in **dgLie**. When there is no risk of confusion we drop the subscripts on σ_{q-1} and τ_q.

Lemma 8.10.2. *Let $q > r+1$ and consider the map of dg Lie algebras $i : \mathbb{L}^g S(q-1) \to \mathbb{L}^g D(q)$ induced by the inclusion $S(q-1) \to D(q)$. This map is a cofibration in* **dgLie**$_r$.

Proof. We must show that i has the LLP with respect to all acyclic fibrations. It suffices to solve the lifting problem at the level of dg vector spaces since $\mathbb{L}^g$ is a left adjoint to the forgetful functor. Consider a lifting problem in **dgVect**:

$$\begin{array}{ccc} S(q-1) & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle p}\ \sim \\ D(q) & \xrightarrow[\beta]{} & Y \end{array}$$

where p is an acyclic fibration in **dgLie**$_r$. The solution is a diagram chase, but since the result is key to proving the model category axioms we include it. There are only two non-trivial levels which we must complete in order to get the dg map γ, namely levels q and $q-1$. Suppressing some of the indices the diagrams look as follows

$$\begin{array}{ccc} \mathbb{Q}\sigma & \xrightarrow{\alpha} & X_{q-1} \\ {\scriptstyle id}\downarrow & \overset{\gamma_{q-1}}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathbb{Q}\sigma & \xrightarrow[\beta]{} & Y_{q-1} \end{array} \quad \text{(degree } q-1\text{)}, \qquad \begin{array}{ccc} 0 & \xrightarrow{\alpha} & X_q \\ \downarrow & \overset{\gamma_q}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathbb{Q}\tau & \xrightarrow[\beta]{} & Y_q \end{array} \quad \text{(degree } q\text{)}.$$

By the commutativity of the left-hand square we must have

$$\gamma_{q-1}(\sigma) = \alpha(\sigma).$$

which indeed does make the filled left-hand square commute, i.e. $p(\alpha(\sigma)) = \beta(\sigma)$. Since

$$d\beta(\tau) = \beta(d\tau) = \beta(\sigma) = p(\alpha(\sigma))$$

we see that $p(\alpha(\sigma))$ is a boundary. Since p induces an isomorphism on homology and since $\alpha(\sigma)$ is a cycle, this implies that $\alpha(\sigma)$ is a boundary, say $\alpha(\sigma) = \partial x$.

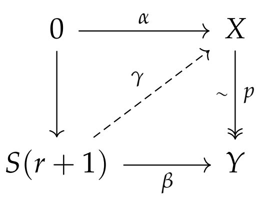
Then $\beta(\tau) - p(x)$ is a cycle (since $\partial p(x) = p(\alpha(\sigma)) = \beta(\sigma)$). Since p induces an isomorphism on homology we can find $x' \in Z_q(X)$, a q-cycle in X, corresponding to $\beta(\tau) - p(x)$. Then there is some $\tilde{y} \in Y_{q+1}$ such that

$$\beta(\tau) - p(x) = p(x') + \partial \tilde{y}.$$

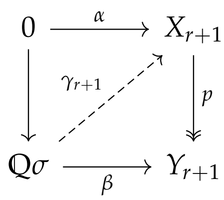
Since p is surjective in degree $q+1$ there is some $\tilde{x} \in X_{q+1}$ such that $p(\tilde{x}) = \tilde{y}$. One can now check that setting $\gamma(\tau) = x + x' + \partial \tilde{x}$ completes the proof. □

Lemma 8.10.3. *The map* $0 \to \mathbb{L}^g S(r+1)$ *is a cofibration in* $\mathbf{dgLie}_r$.

Proof. As before, since $\mathbb{L}^g$ is left adjoint to the forgetful functor to dg vector spaces, it suffices to solve the following lifting problem



In degree $r+1$ the problem looks as follows.



Now since $Y_r = 0$, $\beta(\sigma) \in Y_{r+1}$ is a cycle. Since p induces an isomorphism on homology, there is some $x \in Z_{r+1}(X)$ such that $\beta(\sigma) = p(x) + d\tilde{y}$ for some $\tilde{y} \in Y_{r+2}$. By assumption, p is surjective in degree $r+2$ so there is some $\tilde{x} \in X_{r+2}$ with $p(\tilde{x}) = \tilde{y}$. One can check that $\sigma \mapsto d\tilde{x} + x$ defines the required map γ. □

Lemma 8.10.4. *For* $q > r+1$ *the map* $0 \to \mathbb{L}^g D(q)$ *has the LLP with respect to all fibrations.*

Proof. The proof is similar to Lemma 8.10.3 but easier. If $p : X \twoheadrightarrow Y$ is a fibration in a given lifting problem then one uses that p_q is surjective to find the relevant element of X_q. The details are left to the reader. □

Definition 8.10.3. Suppose $f : \mathfrak{g} \to \mathfrak{h}$ is a map of dg Lie algebras. We call f **almost free** if $\mathfrak{h}$ is isomorphic to a coproduct (in **dgLie**) of $\mathfrak{g}$ with an almost free dg Lie algebra $\mathbb{L}^g(V)$ (i.e. a dg Lie algebra whose underlying graded Lie algebra is free) in such a way that f is isomorphic to the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g} \amalg \mathbb{L}^g(V)$. (cf. Appendix A.4for the corresponding notion between simplicial objects). We define the n-skeleton $\mathfrak{h}^{(n)}$ of f to be the graded sub Lie algebra of $\mathfrak{h}$ generated by the image $f(\mathfrak{g})$ and all elements from V of degree $\leq n$.

Definition 8.10.4. A dg Lie algebra $\mathfrak{g}$ is called **almost free** if the map $0 \to \mathfrak{g}$ is an almost free map. Thus $\mathfrak{g}$ is almost free if $\mathfrak{g}$ is isomorphic (as a graded Lie algebra) to $\mathbb{L}^g(V)$ for some dg vector space V.

Lemma 8.10.5. *Let $f : \mathfrak{g} \to \mathfrak{h}$ be an almost free map of dg Lie algebras (cf. Definition 8.10.3). The n-skeleton of f is obtained from the $n-1$-skeleton of f by attaching n-cells. More precisely, for each $n \geq 0$ there is a pushout square*

$$\begin{array}{ccc} \amalg_\alpha \mathbb{L}^g S(n-1) & \longrightarrow & \mathfrak{h}^{(n-1)} \\ \big\downarrow & & \big\downarrow \\ \amalg_\alpha \mathbb{L}^g D(n) & \longrightarrow & \mathfrak{h}^{(n)} \end{array} .$$

We will not prove this lemma here, but the strategy one uses is a modification of the proof we gave of Proposition 8.9.1.
Since cofibrations in $\mathbf{dgLie}_r$ are defined by their lifting property it follows from Lemma 8.10.5 along with Lemma 8.10.2 and Lemma 8.10.3 that almost free maps are cofibrations.

Lemma 8.10.6. *Any map f in $\mathbf{dgLie}_r$ may be factored into $f = pi$ where i is an almost free map and p is an acyclic fibration.*

Corollary 8.10.1. *Any map f in $\mathbf{dgLie}_r$ may be factored into $f = pi$ where i is a cofibration and p is an acyclic fibration.*

Proof. This is an immediate corollary of Lemma 8.10.6 and the comments preceding it. □

Proposition 8.10.1. *A map f in $\mathbf{dgLie}_r$ is a cofibration if and only if it is a retract of an almost free map.*

Proof. Suppose f is a cofibration. Lemma 8.10.6 provides a factorization $f = pi$ where i is an almost free map and p is an acyclic fibration. Thus f has the LLP with respect to p and so is a retract of i by the retract argument (Lemma 2.3.1)
The converse is clear since almost free maps are cofibrations. □

Lemma 8.10.7. *Any map $f : \mathfrak{g} \to \mathfrak{h}$ in $\mathbf{dgLie}_r$ may be factored as $f = pi$ where i is an acyclic cofibration and p is a fibration.*

Proof. An element $x \in \mathfrak{h}_q$ with $q > r+1$ corresponds to a morphism $\mathbb{L}^g D(q) \to \mathfrak{h}$. Thus letting V be a direct sum of $D(q)$, one copy for each element $x \in \mathfrak{h}_q$ (for all $q > r+1$), we get a dg Lie map

$$p : \mathfrak{g} \amalg \mathbb{L}^g(V) \longrightarrow \mathfrak{h}$$

which is surjective in degrees $> r+1$, i.e. p is a fibration. The inclusion map $i : \mathfrak{g} \hookrightarrow \mathfrak{g} \amalg \mathbb{L}^g(V)$ is almost free, hence by Proposition 8.10.1 it is a cofibration.

We must show that i is a weak equivalence, i.e. induces an isomorphism on homology. This follows from the observation that $H(D(q)) = 0$ and since homology commutes with the free functor (as can be shown using the Künneth isomorphism, see [Qui69, Part I, Proposition 4.5]). Together these facts imply

$$H(\mathfrak{g} \amalg \mathbb{L}^g(V)) \cong H(\mathfrak{g}) \amalg H(\mathbb{L}^g(V)) \cong H(\mathfrak{g}) \amalg \mathbb{L}^g H(V) \cong H(\mathfrak{g})$$

since V is a direct sum of $D(q)$'s for various q. This completes the proof. □

To finish the proof of the model category axioms for $\mathbf{dgLie}_r$ we must verify the final lifting axiom.

Lemma 8.10.8. *Any acyclic cofibration f in $\mathbf{dgLie}_r$ has the LLP with respect to fibrations.*

Proof. From the proof of Lemma 8.10.7 we can factor f as $f = pi$ where p is a fibration and i is an almost free map. By the 2-out-of-3 property, p is an acyclic fibration, thus f has the LLP with respect to p, and so is a retract of i. Now i is a pushout

$$\begin{array}{ccc} 0 & \longrightarrow & \mathfrak{g} \\ \big\downarrow & & \big\downarrow{\scriptstyle in_0} \\ \mathbb{L}^g(V) & \longrightarrow & \mathfrak{g} \amalg \mathbb{L}^g(V) \end{array} .$$

and $0 \to \mathbb{L}^g(V)$ is itself a coproduct of maps $0 \to \mathbb{L}^g D(q)$ for various $q > r + 1$. Lemma 8.10.4 shows that $0 \to D(q)$ has the LLP with respect to fibrations. Thus so does $0 \to \mathbb{L}^g(V)$, and further so does $i : \mathfrak{g} \to \mathfrak{g} \amalg \mathbb{L}^g(V)$. Since f is a retract of i, f too has the LLP with respect to fibrations. □

This completes the proof of Theorem 8.10.1.

Remark 8.10.1. We did not use the Lie bracket in the above proof [check this!]. It therefore seems that the same structure would work for any pointed complete and cocomplete category $\mathcal{C}$ equipped with a "forgetful" functor $U : \mathcal{C} \to \mathbf{dgVect}$ having a left adjoint $L : \mathbf{dgVect} \to \mathcal{C}$.

Using Proposition 2.4.5 we can draw the following conclusion about the associated homotopy category.

Theorem 8.10.2. *The homotopy category* $\mathrm{Ho}(\mathbf{dgLie}_r)$ *is equivalent to the category of almost free dg Lie algebras with morphisms given by homotopy equivalence classes of dg Lie algebra morphisms.*

8.11 Connection with $\mathbf{sLie}_r$.

In this section we will prove that the adjoint functors $N^* \dashv N$ form a Quillen equivalence. As a result they induce an equivalence of categories between $\mathrm{Ho}(\mathbf{sLie}_r)$ and $\mathrm{Ho}(\mathbf{dgLie}_r)$.

Proposition 8.11.1. *The pair $(N*, N)$ of adjoint functors form a Quillen equivalence.*

$$\mathbf{sLie}_r \underset{N^*}{\overset{N}{\rightleftarrows}} \mathbf{dgLie}_r$$

when $r \geq 0$. In particular they induce an adjoint equivalence of categories $\mathrm{Ho}(\mathbf{sLie}_r) \cong \mathrm{Ho}(\mathbf{dgLie}_r)$.

Proof. By Proposition 8.8.1 the two functors are adjoint with N^* left adjoint. By Theorem 8.10.2 we know that the cofibrant dg Lie algebras in $\mathbf{sLie}_r$ are the r-reduced free simplicial Lie algebras. All r-reduced simplicial Lie algebras are clearly fibrant. Suppose L is a cofibrant r-reduced dg Lie algebra and $\mathfrak{g}$ is an r-reduced simplicial Lie algebra. Given a weak equivalence $f : N^*L \underset{\sim}{\to} \mathfrak{g}$ in $\mathbf{sLie}_r$, the adjoint of f is the composition

$$L \xrightarrow{\eta} NN^*L \xrightarrow{Nf} N\mathfrak{g}.$$

The map η is a weak equivalence since $r \geq 0$ and L is free so we can apply Theorem 8.9.1). The map $N(f)$ is a weak equivalence since the homology of this map computes the induced map on *homotopy* by f, i.e. using Moore's formula for the homotopy groups of simplicial groups.
Conversely, suppose $g : L \underset{\sim}{\to} N\mathfrak{g}$ is a weak equivalence in $\mathbf{dgLie}_r$, where L is a free dg Lie algebra. We must show that the adjoint $N^*L \to \mathfrak{g}$ is a weak equivalence in $\mathbf{sLie}_r$. Again, this reduces to showing that the map $NN^*L \to N\mathfrak{g}$ is a quasi-isomorphism since $H \circ N(f) = \pi(f)$. Consider the commutative diagram

$$\begin{array}{ccccc} NN^*L & \xrightarrow{NN^*g} & NN^*N\mathfrak{g} & \xrightarrow{N\varepsilon} & N\mathfrak{g} \\ \uparrow{\scriptstyle \eta\,\sim} & & & \nearrow{\scriptstyle g\,\sim} & \\ L & & & & \end{array}$$

By the 2-out-of-3 property for weak equivalences in $\mathbf{sSet}_r^{\mathrm{Q}}$ we see that the adjoint $N^*L \to \mathfrak{g}$ is indeed a weak equivalence in $\mathbf{sLie}_r$. Thus, (N^*, N) form a Quillen equivalence. The second claim follows from Theorem 2.6.2. □

Appendix A

Appendix

A.1 Minimal fibrations

Here we define minimal Kan complexes and minimal fibrationa. We follow [GJ99, chap. 1, sec. 10] and [May67, chap. 10].

Minimal fibrations and minimal Kan complexes are "rigidifications" or "strictifications" of the usual concepts, fibration and Kan complex, respectively. In Section 3.2 we uses the theory of minimal fibrations to prove the model category axioms for $\mathbf{sSet}_r^{\mathbb{Q}}$.

Minimal Kan complexes

Recall that two n-simplicies $x, y \in K_n$ of a simplicial complex K are said to be *homotopic*, denoted $x \simeq y$, if $d_i x = d_i y$ for all $0 \leq i \leq n$ and there is some $(n+1)$-simplex $z \in K_{n+1}$ such that $d_n z = x$, $d_{n+1} z = y$ and $d_i z = s_{n-1} d_i x = s_{n-1} d_i y$ for $0 \leq i < n$. When K is a Kan complex this relation is an equivalence relation.

Definition A.1.1. A Kan complex K is said to be **minimal** if whenever two simplicies of K are homotopic they are in fact equal.

The following is a very concrete characterization of minimality.

Lemma A.1.1. *Let K be a Kan complex. Then K is minimal if and only if for all n-simplices x and y, if $d_i x = d_i y$ for all $i \neq k$ then $d_k x = d_k y$.*

Proof. "$\Rightarrow$" Suppose K is minimal and suppose $d_i x = d_i y$ for all $i \neq k$ for $x, y \in K_{n+1}$. By minimality it suffices to show that $d_k x \simeq d_k y$. There are two cases. If $k \leq q$ then the family $s_n d_0 x, \cdots, \widehat{s_n d_k x}, \cdots, s_n d_n x, x, y$ in K_{n+1} is compatible and so the Kan extension condition yields a $z \in K_{n+2}$. Now one can show that $d_k z$ is a homotopy from $d_k x$ to $d_k y$.

If $k = n+1$, one instead uses the family $s_{n-1}d_ix$ $(0 \leq i < n)$ and x, y in K_{n+1} which is compatible yielding $z \in K_{n+2}$ such that d_{n+2} is a homotopy between $d_{n+1}x$ and $d_{n+1}y$.
"$\Leftarrow$" Suppose K satisfies the second condition. Suppose $x \simeq y$ for some $x, y \in K_n$. Then there is some $z \in K_{n+1}$ such that $d_{n+1}z = y$, $d_nz = x$ and $d_iz = s_{n-1}d_ix = s_{n-1}d_iy$ for $i < n$. Now for $i < n$ we have $d_iz = s_{n-1}d_ix = d_is_nx$. Also $d_nz = d_ns_nx$. Thus by the second condition applied to z and s_nx it follows that $d_{n+1}z = d_{n+1}s_nx$ i.e. $y = x$. □

Minimal Fibrations

Definition A.1.2. Let $p : E \to B$ be a simplicial map. If $x, y \in E_n$ we say that x is p-**homotopic** to y, written $x \simeq_p y$, if $x \simeq y$ and there is some $z \in E_{n+1}$ such that $z : x \simeq y$ and $p(z) = s_np(x)$.

If p is a Kan fibration then $\simeq_p$ is an equivalence relation.

Definition A.1.3. A Kan fibration $p : E \to B$ is said to be **minimal** if $x \simeq_p y$ implies $x = y$ for $x, y \in E$.

The following proposition says that, up to homotopy, we can always assume that a given Kan fibration is minimal.

Proposition A.1.1. *([GJ99, chap. 1, Proposition 10.3]) Let $p : E \to B$ be a Kan fibration. Then there is a minimal Kan fibration $p' : E' \to B$ which is a strong fibre-wise deformation retract of p.*

A.2 Topological spaces

Here we briefly introduce the category of spaces. Although it plays only a minor role in the thesis, the assumption that we are working in a "convenient category of spaces" is necessary. The category of k-spaces contains a lot of spaces, but is "small enough" to allow for very nice categorial properties. This prioritization of categorial properties over well-behaved objects reflects the general philosophy of many parts of this thesis.

k-Spaces

Definition A.2.1. Suppose X is a topological space. A subset $U \subseteq X$ is called **compactly open** if for every continuous map $f : K \to X$ where K is compact Hausdorff, $f^{-1}(U)$ is open in K.

Definition A.2.2. A k-**space** (or **Kelley space**) is topological space X where every compactly open subset is open.

Let **kTop** denote the full subcategory of **Top** on the k-spaces.

Proposition A.2.1. *([Hov99, Proposition 2.4.22]) The inclusion functor* $i : \mathbf{kTop} \hookrightarrow \mathbf{Top}$ *has a right adjoint* $k : \mathbf{Top} \to \mathbf{kTop}$ *called the "associated k-space".*

If X is a topological space then kX is the set X with the following topology: a subset U is open in X if and only if U is compactly open.

Proposition A.2.2. *([Hov99, Proposition 2.4.22]) The category* **kTop** *has the following structure and properties.*

(i) **kTop** *is complete and cocomplete.*
(ii) **kTop** *contains the geometric realization of all simplicial sets.*
(iii) **kTop** *is Cartesian closed; The internal mapping space is given by the k-topology associated to the compact-open topology. X and Y are objects of* **kTop** *and K is a simplicial set then there is a natural isomorphism of set*

$$\mathrm{Hom}_{\mathbf{kTop}}(X \times |K|, Y) \cong \mathrm{Hom}_{\mathbf{kTop}}(X, Y^{|K|})$$

where $Y^{|K|}$ *carries the k-topology associated to the compact-open topology.*

One important motivation for restricting to a subcategory of **Top** is to get the following result.

Theorem A.2.1. *([Hov99, Lemma 3.2.4]) Let X and Y be simplicial sets. Then the map* $|X \times Y| \to |X| \times_k |Y|$ *is a homeomorphism. In fact, the geometric realization* $|\cdot| : \mathbf{sSet} \to \mathbf{kTop}$ *preserves all finite limits.*

A.3 Completions

The purpose of this section is to introduce, without proofs, completions of algebraic structures such as groups, rings, and Hopf algebras. The category of *complete* Hopf algebras plays an important role in Quillen's rational homotopy theory. For the basic theory we follow [AM69, chap. 10].

Abelian groups

We aim to study completions of Hopf algebras. A Hopf algebra is in particular a ring, and a ring is in particular an Abelian group. So let us start with completions of Abelian groups.

Suppose we are given a sequence of Abelian groups and group homomorphisms,

$$B_0 \xleftarrow{\theta_1} B_1 \xleftarrow{\theta_2} B_2 \xleftarrow{\theta_3} B_3 \xleftarrow{\theta_1} \cdots \xleftarrow{\theta_n} B_n \xleftarrow{\theta_{n+1}} \cdots$$

forming a special diagram in the category of Abelian groups, a so-called *inverse system*. The limit of this diagram is called the *inverse limit* and denoted $\varprojlim_n B_n$. This limit may be calculated explicitly as the underlying set

$$\varprojlim_n B_n = \left\{ (x_n)_{n\in\omega} \in \prod_{n\in\omega} B_n \mid \theta_{n+1}x_{n+1} = x_n \right\}$$

and equipped with point-wise group operation and the obvious coordinate projections.
Suppose A is a fixed Abelian group and

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \supseteq A_n \supseteq \cdots$$

is a decreasing sequences of subgroups. The induced sequence

$$0 = A/A \xleftarrow{\theta_1} A/A_1 \xleftarrow{\theta_2} A/A_2 \xleftarrow{\theta_3} A/A_3 \xleftarrow{\theta_4} \cdots \xleftarrow{\theta_n} A/A_n \xleftarrow{\theta_{n+1}} \cdots$$

forms an inverse system. The inverse limit $\varprojlim_n A/A_n$ is called the **completion of** A with respect to the filtration $\{A_n\}_{n\in\omega}$, it is denoted $\widehat{A}$. A homomorphism $f : A \to B$ between filtered groups is said to **respect** the filtration if $f(A_n) \subseteq B_n$ for all n.
By definition completion is functorial, i.e. given a filtration respecting homomorphism $f : A \to B$ between filtered groups induces a homomorphism $\widehat{f} : \widehat{A} \to \widehat{B}$. We have $\widehat{id} = id$ and $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.

Remark A.3.1. This definition is purely algebraic (i.e. taking some limit in the category of Abelian groups). One can define $\widehat{A}$ as a topological completion as well. Briefly this is done by giving A the topology defined by deeming the sequence $\{A_n\}_{n\in\omega}$ to be a fundamental system of neighborhoods of $0 \in A$. This defines a topology making A into a topological group. We can now define Cauchy sequences as usual, and define $\widehat{A}$ to be the set of Cauchy sequences in A modulo the equivalence relation saying that two sequences are equivalent if and only if their difference converges to 0. Then the two definitions of $\widehat{A}$ agree. In the topological terminology the functoriality may be states as follows: if $f : A \to B$ is a continuous homomorphism, then f induces a continuous homomorphism $\widehat{f} : \widehat{A} \to \widehat{B}$.

Proposition A.3.1. *The functor* $\varprojlim : \mathbf{Ab}^{(\omega^{op})} \to \mathbf{Ab}$ *is left-exact. If*

$$0 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 0$$

is an exact sequence in $\mathbf{Ab}^{(\omega^{op})}$ *such that the morphisms* θ_n^A *in* $\{A_n\}$ *are all surjective, then*

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$$

is exact.

Corollary A.3.1. *Let A be an Abelian group with a decreasing sequence of subgroups $\{A_n\}_{n\in\omega}$ as above. Then $\widehat{A}_n$ is a subgroup of $\widehat{A}$ and*

$$\widehat{A}/\widehat{A}_n \cong \widehat{A/A_n} \cong A/A_n.$$

As a further corollary, the completion is complete.

Corollary A.3.2. *Let A be an Abelian group with a decreasing sequence of subgroups. Then $\widehat{\widehat{A}} \cong \widehat{A}$.*

Example A.3.1. Let $A = k[x]$ be the ring of polynomials over some field k and $A_n = (x)^n = (x^n)$. Then $\widehat{A} = k[[x]]$ is the additive group of formal power series over k.

Example A.3.2. Let $A = \mathbb{Z}$ be the ring integers, p some prime and $A_n = p^n\mathbb{Z}$. Then $\widehat{A} = \mathbb{Z}_p$ is the additive group of p-adic integers.

In general, if A is an Abelian group with a filtration $\{A_n\}_{n\in\omega}$ then the map $\varphi : A \to \widehat{A}$ sending an element $a \in A$ to the constant sequence $(a, a, a, ...) \in \widehat{A}$ is a group homomorphism.

Definition A.3.1. Let A be an Abelian group with a filtration $\{A_n\}_{n\in\omega}$. We say that A is **complete** (with respect to the given filtration) if the map $\varphi : A \to \widehat{A}$ is an isomorphism.

Completeness amounts to the following two conditions.

(i) (Injectivity of φ) If, $\bigcap_n A_n = 0$.
(ii) (Surjectivity of φ) If, for any sequence $(a_1, a_2, \dots)$ where $a_{n+1} \equiv a_n \pmod{A_n}$ $(n \geq 1)$, there exists $a \in A$ with $a \equiv a_n \pmod{A_n}$ for all n.

Vector space completions

In this section we define and study filtered vector spaces and their completions. We define colimits and tensor products of filtered vector spaces and define associated graded vector spaces. This will give the linear algebra framework which we can draw on for the discussion of complete Hopf algebras.
All vector spaces will be over a field k of characteristic zero.

Definition A.3.2. A **filtered vector space** M over k is a vector space M equipped with a filtration by subspaces

$$M = F_0M \supseteq F_1M \supseteq F_2M \supseteq \cdots .$$

A **morphism** of filtered vector spaces is a linear map $f : M \to N$ which preserves the filtration in the sense that $f(F_nM) \subseteq F_nN$ for all $n \in \omega$.

This defines a category of filtered vector spaces denoted $\mathbf{Vect}^{filt}$. This category inherits many of the properties of the category **Vect**.

Example A.3.3. The field k is filtered by $F_0 k = k$ and $F_n k = 0$ for $n > 0$.

Example A.3.4. Given a k-algebra A and an ideal $I \subseteq A$ the powers of I form a filtration $A \supseteq I \supseteq I^2 \supseteq I^3 \supseteq \cdots$. This filtration is called the **I-adic** filtration.

Proposition A.3.2. *The category* $\mathbf{Vect}^{filt}$ *is cocomplete.*

Proof. Coproducts: Given a family $\{M_\alpha\}_{\alpha \in I}$ of filtered vector spaces, the direct sum $\bigoplus_\alpha M_\alpha$ may be equipped with the filtration $F_n \bigoplus_\alpha M_\alpha = \bigoplus_\alpha F_n M_\alpha$. The inclusion maps $i_\alpha : M_\alpha \to \bigoplus_\alpha M_\alpha$ are filtration preserving. This defines a coproduct in $\mathbf{Vect}^{filt}$.
Cokernels: Given a morphism $f : M \to N$ of filtered vector spaces, the image $f(M) \subseteq N$ has a filtration defined by $F_n f(M) = f(M) \cap F_n N$. The quotient (taken in **Vect**), $N/f(M)$ may then be equipped with the filtration given by $F_n(N/f(M)) = (F_n)/(F_n M)$. This filtered vector space, together with the projection map $N \to N/f(M)$ defines a cokernel in $\mathbf{Vect}^{filt}$.
Since $\mathbf{Vect}^{filt}$ has cokernels and small coproducts it is cocomplete. □

Definition A.3.3. The **completion** of a filtered vector space M is the limit $\widehat{M} = \varprojlim_n M/F_n M$. The quotient maps $q_n : M \to M/F_n M$ induce a natural map $q : M \to \widehat{M}$. A filters vector space M is said to be **complete** if q is an isomorphism.

The completion $\widehat{M}$ of a filtered vector space is again filtered, by

$$F_n \widehat{M} = \ker(\widehat{M} \to M/F_n M)$$

where the maps $\widehat{M} \to M/F_n M$ are the projection maps defining $\widehat{M}$ as a limit. This defines a functor $\widehat{()} : \mathbf{Vect}^{filt} \to \mathbf{Vect}^{filt}$. In fact the completed vector space $\widehat{M}$ is complete with respect to this filtration, as the following proposition states.

Proposition A.3.3. *The completion functor* $\widehat{()} : \mathbf{Vect}^{filt} \to \mathbf{Vect}^{filt}$ *is idempotent, in the sense that* $\widehat{\widehat{M}} \cong \widehat{M}$.

So the completion functor $\widehat{()}$ takes values in the category of complete vector spaces. This category will be denoted $\mathbf{Vect}^{comp}$, it is the full subcategory of $\mathbf{Vect}^{filt}$ generated by those filtered vector spaces which are complete. The functor $\widehat{()} : \mathbf{Vect}^{filt} \to \mathbf{Vect}^{comp}$ is left adjoint to the inclusion functor $\mathbf{Vect}^{comp} \to \mathbf{Vect}^{filt}$, thus $\mathbf{Vect}^{comp}$ is a reflective subcategory of $\mathbf{Vect}^{filt}$.

Corollary A.3.3. *The category* $\mathbf{Vect}^{comp}$ *is cocomplete.*

Proof. Colimits may be computed first in $\mathbf{Vect}^{filt}$ and then completed. □

Complete tensor products

Given filtered vector space M and N, the tensor usual product $M \otimes N$ (taken over k) admits a filtration given by

$$F_n(M \otimes N) = \bigoplus_{r+s=n} F_r M \otimes F_s N$$

for each n. Even if M and N are both complete, their tensor product need not be.

Definition A.3.4. The **tensor product** of complete filtered vector spaces M and N, denoted $M \widehat{\otimes} N$ is the completion of their tensor product as filtered vector spaces, i.e.

$$M \widehat{\otimes} N = \widehat{M \otimes N} = \lim_n (M \otimes N)/F_n(M \otimes N).$$

Proposition A.3.4. *If M and N are filtered vector spaces then the morphism $M \otimes N \to \widehat{M} \otimes \widehat{N} \to \widehat{M} \widehat{\otimes} \widehat{N}$ extends to an isomorphism $\widehat{M \otimes N} \xrightarrow{\cong} \widehat{M} \widehat{\otimes} \widehat{N}$.*

A.4 Almost free morphisms

The purpose of this section is to introduce *almost free maps* and *almost free objects* and prove some elementary properties. An early reference for this material is [Kan57].

Remark A.4.1. The terminology does not seem to be completely standard. What Quillen and Kan call "free" we shall call "almost free", following [GJ99].

Definitions

Let $\mathcal{C}$ be a cocomplete category equipped with a cocontinuous functor $F : \mathrm{Set} \to \mathcal{C}$. Let Δ_{surj} be the category of finite ordinals and *surjective* order-preserving maps. We shall study morphisms in $s\mathcal{C}$, the category of simplicial objects in $\mathcal{C}$. The following definition is a direct generalization of the notion of almost free maps between simplicial groups as it appears in [GJ99, chap. 5, sec. 1].

Definition A.4.1. A simplicial map $f : A \to B$ in $s\mathcal{C}$ is said to be F-**almost free** if there is a functor $X : \Delta_{surj}^{op} \to \mathrm{Set}$ and a collection $\{\theta_n\}_{n \in \omega}$ of isomorphisms,

$$A_n \amalg F(X_n) \xrightarrow[\cong]{\theta_n} B_n$$

satisfying the following compatibility conditions: for each n the following diagram commutes

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & B_n \\ & {\scriptstyle in_{A_n}} \searrow & \nearrow {\scriptstyle \theta_n}\, \cong \\ & A_n \amalg F(X_n) & \end{array}$$

and for each $t : \mathbf{m} \twoheadrightarrow \mathbf{n}$ in Δ_{surj} the following diagram commutes

$$\begin{array}{ccc} A_n \amalg F(X_n) & \xrightarrow{\theta_n} & B_n \\ {\scriptstyle t^* \amalg F(X(t))}\big\downarrow & & \big\downarrow{\scriptstyle t^*} \\ A_m \amalg F(X_m) & \xrightarrow{\theta_m} & B_m. \end{array}$$

Definition A.4.2. A simplicial object B is said to be F-**almost free** if the initial map $0 \to B$ is F-almost free.

Remark A.4.2. In the applications $\mathcal{C}$ will be some category of algebraic objects (e.g. groups or Lie algebras) and F will be the free functor, left adjoint to the forgetful functor. In this case, we suppress the F and simply speak of **almost free** maps and **almost free** objects. Then the above definition may be roughly phrased as follows: a map $f : A \to B$ is almost free if there is a choice of sets X_n which are stable under degeneracies and such that the coproduct of A_n with the freely generated structure on X_n is isomorphic to B_n.

Basic Properties

Let $(\mathcal{C}, F)$ be as above. Since F preserves all colimits we have the following closure properties for the class of F-almost free maps.

Proposition A.4.1. *The class of F-almost free maps is closed under the following operations.*

(i) Isomorphism: *If $f : A \to B$ is F-almost free and if $f' : A' \to B'$ is an isomorphic map, then f' is F-almost free.*

(ii) Composition: *If $f : A^0 \to A^1$ and $g : A^1 \to A^2$ are F-almost free then $g \circ f : A^0 \to A^2$ is F-almost free.*

(iii) Sequential composition: *Let $A : \omega \to s\mathcal{C}$ be an ω-sequence, i.e. a diagram*

$$A^0 \xrightarrow{f^0} A^1 \xrightarrow{f^1} \cdots \to A^n \xrightarrow{f^n} A^{n+1} \to \cdots$$

of maps in $s\mathcal{C}$. If each map $A^n \to A^{n+1}$ is F-almost free, then $A^0 \to \mathrm{colim}_n A^n$ is F-almost free.

(iv) Coproducts: *If $f : A \to B$ and $g : A \to B$ are F-almost free then $f \amalg g$ is F-almost free.*

(v) Pushouts: *If $f : A \to B$ is F-almost free and and $g : A \to C$ is some morphism in $s\mathcal{C}$, then the pushout of $f_* : C \to C \amalg_A B$ is F-almost free.*

Proof. (i) : We suppose given an isomorphism

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ \cong\big\downarrow\alpha & & \cong\big\downarrow\beta \\ A' & \xrightarrow{\ f'\ } & B' \end{array}$$

of morphisms. To show f' is F-almost free we can use the same functor X as for f and isomorphisms θ'_n defined by requiring the following diagram be commutative:

$$\begin{array}{ccc} A_n \amalg F(X_n) & \underset{\cong}{\xrightarrow{\ \theta_n\ }} & B \\ \alpha\amalg id\big\downarrow\cong & & \cong\big\downarrow\beta \\ A'_n \amalg F(X_n) & \overset{\theta'_n}{\dashrightarrow} & B'_n \end{array}$$

The compatibility conditions are easily verified.
(ii) : We have maps θ^0, θ^1 and functors X^0 and X^1 witnessing that f and g are F-almost free. Now take the following composite isomorphism

$$A^0_n \amalg F(X^0_n \amalg X^1_n) \cong A^0_n \amalg F(X^0_n) \amalg F(X^1_n) \underset{\cong}{\xrightarrow{\theta^0_n \amalg id}} A^1_n \amalg F(X^1_n) \underset{\cong}{\xrightarrow{\theta^1_n}} A^2_n$$

where the first (natural) isomorphism comes from the cocontinuity of F. Call the composite ψ^1_n. The collection $\{\psi^1_n\}_{n\in\omega}$ satisfies both compatibility conditions.
(iii) : By iterating the construction in part *(ii)* we get a map of ω-sequences (omitting the subscript m):

$$\begin{array}{ccccccccc} A^0 \amalg F(X^0) & \xrightarrow{i^0} & A^0 \amalg F(X^0 \amalg X^1) & \xrightarrow{i^1} & \cdots & \longrightarrow & A^0 \amalg F(\coprod_{k<n} X^k) & \xrightarrow{i^{n-1}} & \cdots \\ \theta^0\big\uparrow & \searrow^{\psi^0:=\theta^0}_{\cong} & & \searrow^{\psi^1}_{\cong} & & & & \searrow^{\psi^{n-1}}_{\cong} & \\ A^0 & \xrightarrow{f^0} & A^1 & \xrightarrow{f^1} & A^2 & \longrightarrow \cdots \longrightarrow & A^n & \longrightarrow & \end{array}$$

where the commutativity follows from the construction in part *(ii)*. Since F is cocontinuous we can compute the colimit of the top sequence as

$$\mathrm{colim}_n(A^0 \amalg F(\coprod_{k<n} X^k)) \cong A^0 \amalg F(\coprod_{k<\omega} X^k).$$

Thus we get an isomorphism

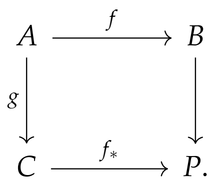
$$\psi^\omega_m : (A^0)_m \amalg F(\coprod_{k<\omega} X^k_m) \underset{\cong}{\longrightarrow} \mathrm{colim}_n(A^n)_m.$$

One can then check that the ψ^ω_m satisfy the compatibility conditions.

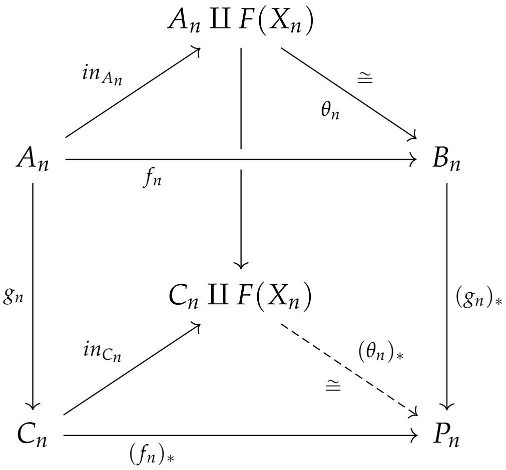
(iv): By assumption there are θ_n, θ'_n and X, X' such that $\theta_n : A \amalg F(X_n) \to B_n$ and $A \amalg F(X'_n) \to B_n$ are isomorphisms. Now take the following composition

$$(A \amalg A) \amalg F(X_n \amalg X'_n) \cong A \amalg A \amalg F(X_n) \amalg F(X'_n) \cong A \amalg F(X_n) \amalg A \amalg F(X'_n) \xrightarrow[\cong]{\theta_n \amalg \theta'_n} B_n \amalg B_n$$

where we have used that F preserves the coproduct. It is straightforward to check that this makes $f \amalg g$ F-almost free.
(v) Consider the following pushout diagram



where f is F-almost free. To show that f_* is F-almost free consider the prism diagram



where the front square is the pushout that we started with, and the back left-hand square is the pushout of in_{A_n} along g_n. The map $(\theta_n)_*$ is induced by $(g_n)_* \circ \theta_n$ and $(f_n)_*$. By the commutativity of the diagram the back right-hand square is also a pushout and so $(\theta_n)_*$ is an isomorphism, as indicated. $\square$

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